ON THE μ-INVARIANT OF THE CYCLOTOMIC DERIVATIVE OF KATZ p-ADIC L-FUNCTION

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Abstract. When the branch character has root number -1, the corresponding anticyclotomic Katz p-adic L-function identically vanishes. In this case, we study the μ-invariant of the cyclotomic derivative of Katz p-adic L-function. As an application, this proves the non-vanishing of the anticyclotomic regulator of a self-dual CM modular form with the root number -1.

Contents

1. Introduction 2
2. Hilbert modular Shimura variety 4
2.1. Setup 4
2.2. Moduli interpretation 5
2.3. CM points 7
2.4. Igusa tower 7
2.5. Tate objects 8
2.6. Deformation theory of an ordinary abelian variety 8
2.7. Geometric Hilbert modular forms 9
2.7.1. Classical Hilbert modular forms 9
2.7.2. p-adic Hilbert modular forms 9
2.7.3. Mod p Hilbert modular forms 10
2.8. Linear independence 10
3. Cyclotomic derivative 11
3.1. Eisenstein series on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$ 12
3.2. Fourier coefficients of Eisenstein series 12
3.3. Choice of the local sections 13
3.4. q-expansion of normalized Eisenstein series 14
3.5. Cyclotomic derivative 15
4. Proof of Theorem A 15
4.1. An outline 16
4.2. The vanishing of an Eisenstein series 17
4.3. A lower bound 18
4.4. An upper bound I 19
4.5. An upper bound II 22
5. Non-vanishing of anticyclotomic regulator 23
References 24

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1. Introduction

Zeta values often enter the p-adic world via p-adic L-functions. One expects that p-adic L-functions are intimately connected with arithmetic. A p-adic L-function can have several variables. In such a case, we may expect that a power series obtained by taking its partial derivative with respect to one of the variables at a specific value of that variable, also has some arithmetic meaning.

Katz p-adic L-function over a totally real field of degree d has $(d+1+\delta)$-variables, where δ is the Leopoldt defect for the totally real field. In this article, by Katz p-adic L-function we mean the projection to the first $(d+1)$-variables. When the branch character is self-dual with the root number -1, the corresponding anticyclotomic Katz p-adic L-function of d-variables identically vanishes. In this article, we study the μ-invariant of the cyclotomic derivative of the Katz p-adic L-function when the branch character is of this type. Following a strategy of Hida, we determine this μ.

Let us introduce some notation. Fix an odd prime p. Let $\mathcal{F}$ be a totally real field of degree d over $\mathbf{Q}$ and $\mathcal{K}$ be a totally imaginary quadratic extension of $\mathcal{F}$. Let D_F be the discriminant of $\mathcal{F}/\mathbf{Q}$. Fix two embeddings $\iota_\infty\colon \overline{\mathbf{Q}} \to \mathbf{C}$ and $\iota_p\colon \overline{\mathbf{Q}} \to \mathbf{C}_p$. Let c denote the complex conjugation on $\mathbf{C}$ which induces the unique non-trivial element of $\mathrm{Gal}(\mathcal{K}/\mathcal{F})$ via ι_∞. We assume the following hypothesis throughout:

(ord) Every prime of $\mathcal{F}$ above p splits in $\mathcal{K}$.

The condition (ord) guarantees the existence of a p-adic CM type Σ i.e. Σ is a CM type of $\mathcal{K}$ such that, p-adic places induced by elements in Σ via ι_p are disjoint from those induced by Σc. Let $\mathcal{K}_\infty^+$ and $\mathcal{K}_\infty^-$ be the cyclotomic $\mathbf{Z}_p$-extension and anticyclotomic $\mathbf{Z}_p^d$-extension of $\mathcal{K}$. Let $\mathcal{K}_\infty = \mathcal{K}_\infty^+\mathcal{K}_\infty^-$ be a $\mathbf{Z}_p^{d+1}$-extension of $\mathcal{K}$. Let $\Gamma^\pm := \mathrm{Gal}(\mathcal{K}_\infty^\pm/\mathcal{K})$ and let $\Gamma = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}) \simeq \Gamma^+ \times \Gamma^-$.

Let $\mathfrak{C}$ be a prime-to-p integral ideal of $\mathcal{K}$. Decompose $\mathfrak{C} = \mathfrak{C}^+\mathfrak{C}^-$, where $\mathfrak{C}^+$ (respectively $\mathfrak{C}^-$) is a product of split primes (respectively ramified or inert primes) over $\mathcal{F}$. Let λ be a Hecke character of infinity type $k\Sigma$, $k>0$ and suppose $\mathfrak{C}$ is the prime-to-p conductor of λ. Associated to this data, a $(d+1)$-variable Katz p-adic L-function $L_{\Sigma,\lambda}(T_1,T_2,...,T_d,S) \in \overline{\mathbf{Z}}_p[\![\Gamma]\!]$ is constructed in [10] and [4] . Here $T_1,...,T_d$ are the anticyclotomic variables and S is the cyclotomic variable. We occasionally abbreviate this function as $L_{\Sigma,\lambda}$. It interpolates critical Hecke L-values $L(0,\lambda\chi)$ as χ varies over certain Hecke characters mod $\mathfrak{C}p^\infty$ (cf. [4, Thm. II]). Let $L_{\Sigma,\lambda}^- \in \overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$ be the anticyclotomic projection obtained by substituting $S=0$.

For each local place v, choose a uniformiser ϖ_v and let $|.|_v$ denote the corresponding absolute value normalised so that $|\varpi_v|_v = |N(\varpi_v)|_l$ and $|l|_l = \frac{1}{l}$, where N is the norm, $v \cap \mathbf{Q} = (l)$ and $l>0$. Let v_p be the p-adic valuation of $\mathbf{C}_p$ normalised such that $v_p(p)=1$. We view it as a function on $\overline{\mathbf{Q}}$ via ι_p. Let N be the norm Hecke character i.e. adelic realisation of the p-adic cyclotomic character. For each v dividing $\mathfrak{C}^-$ and λ as above, the local invariant $\mu_p(\lambda_v)$ is defined by

$$\mu_p(\lambda_v) = \inf_{x \in K_v^\times} v_p(\lambda_v(x)-1). \tag{1.1}$$

Let us also define

$$\mu_p^{'}(\lambda_v) = v_p\left(\frac{\log_p(|\varpi_v|)}{\log_p(1+p)}\right) + \sum_{w \neq v, w|\mathfrak{C}^-} \mu_p(\lambda_w) \geq 0 \tag{1.2}$$

and

$$\mu_p^{'}(\lambda) = \sum_{v|\mathfrak{C}^-} \mu_p(\lambda_v). \tag{1.3}$$

From now on, suppose that λ is self-dual i.e. $\lambda|_{\mathbf{A}_\mathcal{F}^\times} = \tau_{\mathcal{K}/\mathcal{F}}|.|_{\mathbf{A}_\mathcal{F}}$, where $\tau_{\mathcal{K}/\mathcal{F}}$ is the quadratic character associated to $\mathcal{K}/\mathcal{F}$ and $|.|_{\mathbf{A}_\mathcal{F}}$ is the adelic norm. In particular, the global root number of λ is ±1. Now, suppose that the global root number is -1. In view of the functional equation of Hecke L-function, this root number

condition forces all the Hecke L-values appearing in the interpolation property of $L_{\Sigma,\lambda}^{-}$ to vanish. Accordingly, $L_{\Sigma,\lambda}^{-}=0$. This also follows from the functional equation of $L_{\Sigma,\lambda}$ (cf. [4, §5]). The anticyclotomic arithmetic information contained in $L_{\Sigma,\lambda}$ may seem to have disappeared. However, we can look at the cyclotomic derivative

$$L_{\Sigma,\lambda}^{'}=(\frac{\partial}{\partial S}L_{\Sigma,\lambda}(T_1,...,T_d,S))|_{S=0}. \tag{1.4}$$

For any integer k, $L_{\Sigma,\lambda}(T_1,T_2,...,T_d,(1+p)^k-1)$ equals $L_{\Sigma,\lambda N^k}^{-}$ and $\lim((1+p)^{p^n}-1)$ equals zero. Thus, the cyclotomic derivative equals

$$\frac{1}{\log_p(1+p)}(\frac{d}{ds}L_{\Sigma,\lambda N^s}^{-})|_{s=0}\in\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]. \tag{1.5}$$

Note that the factor $\frac{1}{\log_p(1+p)}$ comes from the fact that

$$\lim\frac{(1+p)^{p^n}-1}{p^n}=\log_p(1+p).$$

Here, $\log_p$ is Iwasawa's p-adic logarithm normalised so that $\log_p(p)=0$. In (1.5) and throughout this article our meaning of derivative is the following. Let f be a function from integers to a p-adic domain of characteristic different from p. We define

$$\frac{d}{ds}f(s)|_{s=0}:=\lim\frac{f(p^n)-f(0)}{p^n}. \tag{1.6}$$

Here, lim denotes the p-adic limit. Note that the Leibnitz product rule is valid for this notion of derivative.

The following can be considered as the main result of the article.

Theorem A Let $h_{\mathcal{K}}^{-}:=h_{\mathcal{K}}/h_{\mathcal{F}}$ be the relative class number. Suppose that $p\nmid h_{\mathcal{K}}^{-}\cdot D_{\mathcal{F}}$. Then, we have

$$\mu(L_{\Sigma,\lambda}^{'})=\min_{v|\mathfrak{C}^-}\{\mu_p^{'}(\lambda),\mu_p^{'}(\lambda_v)\}.$$

We now describe the strategy of the proof. Some of the notation used here is not followed in the rest of the article.

We basically follow a strategy of Hida. Let us briefly recall Hida's strategy to determine $\mu(L_{\Sigma,\lambda}^{-})$ (cf. [6]). Suppose that $p\nmid h_{\mathcal{K}}^{-}\cdot D_{\mathcal{F}}$. The starting point is the observation that there are classical Hilbert modular Eisenstein series $(f_{i,\lambda})_i$ such that

$$L_{\Sigma,\lambda}^{-}=\sum_i a_i\circ(f_{i,\lambda}(t)), \tag{1.7}$$

upto an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$, where $f_{i,\lambda}(t)$ is the t-expansion of f_i around a well chosen CM point x with the CM type $(\mathcal{K},\Sigma)$ on the Hilbert modular Shimura variety Sh and a_i is an automorphism of the deformation space of x in Sh. Based on Chai's study of Hecke-stable subvarieties of a Shimura variety, Hida proves the linear independence of $(a_i\circ f_{i,\lambda})_i$ modulo p. It follows that $\mu(L_{\Sigma,\lambda}^{-})=\min_i\mu(f_{i,\lambda}(t))=\min_i\mu(f_{i,\lambda})$. Now, $\mu(f_{i,\lambda})=\mu(f_{i,\lambda}(q))$, where $f_{i,\lambda}(q)$ is the q-expansion of $f_{i,\lambda}$. Thus, the question reduces to the computation of the μ-invariant of the q-expansion. When $\mathfrak{C}^-=1$, this computation can be done quite explicitly. However, when $\mathfrak{C}^-\neq 1$, the computation seems quite complicated. As an alternative, Hsieh constructs certain Hilbert modular Eisenstein series $(\mathbf{f}_{i,\lambda})_i$ whose q-expansion comupuation is a bit simpler than that of $(f_{i,\lambda})_i$ such that the property (1.7) still holds i.e. the power series $L_{\Sigma,\lambda}^{-}$ equals $\sum_i a_i\circ(\mathbf{f}_{i,\lambda}(t))$ upto an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$ (cf. [8]). In [6] and [8], the condition $p\nmid h_{\mathcal{K}}^{-}$ is not needed as otherwise the power series $L_{\Sigma,\lambda}^{-}$ restricted to an explicit finite open cover is still of the form (1.7).

In our case, the root number condition forces that $\mathfrak{C}^{-} \neq 1$. So, we use the later Eisenstein series. Firstly, we show that there are p-adic Hilbert modular forms $(\mathbf{f}_{i,\lambda}^{'})_i$ such that

$$L_{\Sigma,\lambda}^{'} = \sum_i a_i \circ (\mathbf{f}_{i,\lambda}^{'}(t)), \tag{1.8}$$

upto an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$. Basically, $\mathbf{f}_{i,\lambda}^{'}$ is the derivative of $\mathbf{f}_{i,\lambda N^s}$ at $s=0$ (cf. (1.5)). As in Hida's strategy, the question then reduces to the computation of the μ-invariant of q-expansion of $(\mathbf{f}_{i,\lambda}^{'})_i$. However, the expression for the q-expansion coefficients does not seem to be very explcit. The coefficients are the products of certain local Whittaker integrals and derivatives.

When $\mathcal{F}$ equals $\mathbf{Q}$ and λ is the Grössencharacter associated to a CM elliptic curve $E/\mathbf{Q}$ having CM by $\mathcal{O}_K$, the Katz p-adic L-function, $L_{\Sigma,\lambda}$ is the two variable commutative p-adic L-function associated to E. In [12], Rubin proves that $L_{\Sigma,\lambda}$ generates the characteristic ideal of a certain Selmer group associated to $E/\mathcal{K}_\infty$. This two variable main conjecture gives cyclotomic and anticyclotomic main conjectures. When λ has root number -1, both sides of the anticyclotomic main conjecture are zero (cf. [1]). However, in [loc. cit., Thm A] it is shown that $L_{\Sigma,\lambda}^{'}$ generates the characteristic ideal of the torsion part of the anticyclotomic Selmer group times a certain anticyclotomic regulator associated to λ after tensoring with $\mathbf{Q}_p$. In the appendix of [1], Rubin proves the non-vanishing of this anticyclotomic regulator. Thus, $L_{\Sigma,\lambda}^{'}$ is non-trivial. The results of [loc. cit.] have been generalised to self-dual CM modular forms in [2], except the non-vanishing of the anticyclotomic regulator. Theorem A proves the non-vanishing of the anticyclotomic regulator of a CM modular form with the root number -1. This seems to be one of the first instances where the non-vanishing of an Iwasawa theoretic regulator is proven by a modular method (combined with the main conjecture).

When λ is self-dual without a condition on the root number, the method in this paper can be used to study non-triviality of certain higher order anticyclotomic derivatives $L_{\Sigma,\lambda}^{(k)}$ modulo p. As a consequence, we can show that $L_{\Sigma,\lambda} \in \overline{\mathbf{Z}}_p[\![\Gamma]\!]$ is not a polynomial.

It seems likely that $L_{\Sigma,\lambda}^{'}$ generates the characteristic ideal of the torsion part of the anticyclotomic Selmer group upto an anticyclotomic regulator. Another interesting question would be whether a normalisation of $L_{\Sigma,\lambda}^{'}$ interpolates a normalisation of (complex) derivative L-values.

The article is organised as follows. In §2, we recall some facts about Hilbert modular Shimura variety. Basically, we need to state a version of Hida's linear independence result suitable to our settting. This does not seem to be directly stated in [6]. We also recall the notion of the t-expansion of a Hilbert modular form around a CM point which plays an essential role in the article. The reader familiar with [6] can begin with §3. In §3, firstly we recall the construction of Eisenstein series in [9]. Towards the end, the p-adic Hilbert modular forms $(\mathbf{f}_{i,\lambda}^{'})_i$ are constructed. In §4, we prove Theorem A. In §4.1, we firstly give an outline of the proof. In §5, as an application, we prove the non-vanishing of the anticyclotomic regulator in [2].

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2. Hilbert modular Shimura variety

In this section, we recall some facts about Hilbert modular Shimura variety. We end with a certain linear independence of mod p Hilbert modular forms due to Hida. We follow [5], [6] and [9].

2.1. Setup.

In this subsection, we recall a basic setup regarding Hilbert modular Shimura variety.

Let $G = Res_{\mathcal{F}/\mathbf{Q}}GL_2$ and $h_0 : Res_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m \to G_{/\mathbf{R}}$ be the morphism of real group schemes given by

$$a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a + bi \in \mathbf{C}^\times$. Let X be the set of $G(\mathbf{R})$-conjugacy classes of h_0. We have a canonical isomorphism $X \simeq (\mathbf{C} - \mathbf{R})^I$, where I is the set of real places of $\mathcal{F}$. The pair (G, X) satisfies Deligne's axioms for a Shimura variety. It gives rise to a tower $(Sh_K(G, X))_K$ of quasi-projective smooth varieties over $\mathbf{Q}$ indexed by open compact subgroups K of $G(\mathbf{A}^f)$. The tower is endowed with an action of $G(\mathbf{A}^f)$. The pro-algebraic variety $Sh(G, X)/\mathbf{Q}$ is the projective limit of these varieties. The complex points of these varieties are given as follows

$$Sh_K(G,X)(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}^f)/K, Sh(G,X)(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}^f)/\overline{Z(\mathbf{Q})}. \tag{2.1}$$

Here, $\overline{Z(\mathbf{Q})}$ is the closure of the center $Z(\mathbf{Q})$ in $G(\mathbf{A}^f)$ under the *adélic* topology. For $(z, g) \in X \times G(\mathbf{A}^f)$, let $[z, g]$ denote the corresponding point on $Sh(G, X)(\mathbf{C})$.

Let us introduce some notation. Consider $V = \mathcal{F}^2$ as a two dimensional vector space over $\mathcal{F}$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $\langle ., . \rangle : V \times V \to \mathcal{F}$ be the $\mathcal{F}$ bilinear pairing defined by $\langle e_1, e_2\rangle = 1$. Let $\mathcal{L} = Oe_1 \oplus O^*e_2$ be the standard O lattice in V. For a fractional ideal $\mathfrak{b}$ of O, $\mathfrak{b}^* := \mathfrak{b}^{-1}\mathfrak{d}_{\mathcal{F}}^{-1}$. Here, $\mathfrak{d}_?$ denotes the different of $?/\mathbf{Q}$, where ? equals $\mathcal{F}$ or $\mathcal{K}$. Sometimes, we denote $\mathfrak{d}_{\mathcal{F}}$ by $\mathfrak{d}$. For $g \in G(\mathbf{Q})$, $g' := det(g)g^{-1}$. Note that $G(\mathbf{Q})$ has a natural right action on $\mathcal{F}^2$. For $x \in V$, consider the left action $gx := xg'$.

Let h be the set of finite places of $\mathcal{F}$. For $v \in h$,

$$K_v^0 := \{g \in GL_2(\mathcal{F}_v) | g(\mathcal{L} \otimes O_v) = \mathcal{L} \otimes O_v\},\ K_p^0 := \prod_{v|p} K_v^0.$$

From now on, we consider only those open compact subgroups K of $G(\mathbf{A}^f)$ for which K_p equals K_p^0. We say that K is maximal at p if K equals $G(\mathbf{Z}_p) \times K^{(p)}$. Sometimes, by 1 we mean the trivial subgroup.

2.2. **Moduli interpretation.** In this subsection, we describe the moduli functor represented by Hilbert modular Shimura variety.

To describe the functor, we first introduce a certain fibered category. Let Ξ be a finite set of rational primes. Let $\mathbf{Z}_{(\Xi)}$ denote the localisation of $\mathbf{Z}$ at Ξ. Consider the fibered category $\mathcal{A}_K^{\Xi}/SCH_{/\mathbf{Z}_{(\Xi)}}$ as follows. Let $S/\mathbf{Z}_{(\Xi)}$ be a locally Noetherian and connected scheme. Let $\bar{s}$ be a geoemtric point of S. The objects are abelian varieties with real multiplication over S of level K. To be precise, an object $\underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta}^{\Xi})_S$ is a quadruple where

$(rm1)$ A/S is an abelian scheme of dimension d.
$(rm2)$ λ is prime to Ξ polarisation of A/S and $\bar{\lambda} := \{\lambda' \in Hom(A, A^t) \otimes \mathbf{Z}_{\Xi} | \lambda' = \lambda \circ a, a \in O_{(\Xi),+}^{\times}\}$.
Here, $O_{(\Xi),+} := \{a \in O_{(\Xi)} | \sigma(a) > 0, \forall \sigma \in I\}$.
$(rm3)$ $\iota : O \hookrightarrow End_S A \otimes_{\mathbf{Z}} \mathbf{Z}_{(\Xi)}$ is an embedding.
$(rm4)$ $\bar{\eta}^{\Xi} = \eta^{\Xi}K^{\Xi}$ is a $\pi_1(S, \bar{s})$-invariant $K^{(p)}$-orbit of $\mathcal{O}_{\mathcal{K}}$-module isomorphism $\eta^{\Xi} : \mathcal{L} \otimes \mathbf{A}^{f(\Xi)} \simeq H_1(A_{\bar{s}}, \mathbf{A}^{f(\Xi)})$.
Here and henceforth, $\mathbf{A}_?^{f(\Box)}$ denotes the finite *adéles* of $\mathbf{A}_?$ outside a finite set of rational primes $\Box$, of a number field ?. When ? $= \mathbf{Q}$, from the notation we drop the subscript. For $g \in GL_2(\mathbf{A}_{\mathcal{F}}^{f(\Xi)})$, $(\eta^{\Xi}g)(x) := \eta^{\Xi}(gx)$.

We also demand the quadruple to satisfy the following conditions.

$(c1)$ $\forall b \in O$, $\iota(b)^t = \iota(b)$ where ^{t} is the Rosati involution induced by λ.
$(c2)$ We fix an isomorphism $\zeta : \mathbf{A}^f \simeq \mathbf{A}^f(1)$. Thus, we can regard the Weil pairing e^{λ} induced by λ as an $\mathcal{F}$-alternate form $e^{\lambda} : V^{\Xi}(A) \times V^{\Xi}(A) \to \mathfrak{d}_{\mathcal{K}}^{-1} \otimes_{\mathbf{Z}} \mathbf{A}^{f(\Xi)}$. Let e^{η} denote the $\mathcal{F}$-alternate form $e^{\eta}(x, x') := \langle x\eta, x'\eta\rangle$. Then, $e^{\lambda} = ue^{\eta}$ for some $u \in \mathbf{A}_{\mathcal{F}}^{f(\Xi)}$.
$(c3)$ There exists an $O \otimes_{\mathbf{Z}} \mathcal{O}_S$-module isomorphism $LieA \simeq O \otimes_{\mathbf{Z}} \mathcal{O}_S$, locally under the Zariski topology of S.

Let $\underline{\mathcal{A}} = (A, \bar{\lambda}, \iota, \bar{\eta}^{\Xi})$ and $\underline{\mathcal{A}'} = (A', \bar{\lambda}', \iota', \bar{\eta}'^{\Xi})$. We define

(mor) $Mor_{\mathcal{A}_K^{\Xi}}(\underline{\mathcal{A}}, \underline{\mathcal{A}'}) := \{f \in Hom_O(A, A') | f^*\bar{\lambda}' = \bar{\lambda}, f \circ \bar{\eta}'^{\Xi} = \bar{\eta}^{\Xi}\}$.

We say that $\underline{\mathcal{A}} \sim \underline{\mathcal{A}'}$ (resp. $\simeq$) if there exists prime to Ξ isogeny (resp. isomorphism) in $Mor_{\mathcal{A}_K^{\Xi}}(\underline{\mathcal{A}}, \underline{\mathcal{A}'})$.

Consider the functor

$$\mathcal{E}_K^{\Xi} : SCH/\mathbf{Z}_{(\Xi)} \to SETS$$
$$S \mapsto \{\underline{\mathcal{A}} \in \mathcal{A}_K^{\Xi}(S)\}/\sim .$$

Also, let us consider the functor

$$\mathcal{C}_K^{\Xi} : SCH/\mathbf{Z}_{(\Xi)} \to SETS$$
$$S \mapsto \{\underline{\mathcal{A}} \in \mathcal{A}_K^{\Xi}(S) | \eta^{\Xi}(\mathcal{L} \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}) = H_1(A_{\bar{s}}, \hat{\mathbf{Z}})\}/\simeq .$$

In [5, §4.2], it is shown that $\mathcal{E}_K^{\Xi} \simeq \mathcal{C}_K^{\Xi}$.

Let us first consider the case $\Xi = \emptyset$. Accordingly, in the above notation we drop Ξ.

Theorem 2.1. *(Shimura-Deligne) The functor $\mathcal{E}_1$ is represented by $Sh(G, X)/\mathbf{Q}$. When K is small, $\mathcal{E}_K$ is represented by $Sh_K(G, X) = Sh(G, X)/K$ (cf.* [5, §4.2]*).*

Let $\underline{\mathcal{A}}_{K,univ}$ be the universal object.

Now, let us consider the case $\Xi = \{p\}$.

Theorem 2.2. *(Kottwitz) The functor $\mathcal{E}_1^{(p)}$ is represented by $Sh^{(p)}(G, X)/\mathbf{Z}_{(p)}$. Moreover,*

$$Sh^{(p)}(G, X) \times \mathbf{Q} \simeq Sh(G, X)/G(\mathbf{Z}_p)_{/\mathbf{Q}}.$$

When K is small and maximal at p, $\mathcal{E}_K^{(p)}$ is represented by $Sh_K^{(p)}(G, X) = Sh^{(p)}(G, X)/K$ (cf. [5, §4.2.1]*).*

Let $\underline{\mathcal{A}}_{K,univ}^{(p)}$ be the universal object.

Let K be sufficiently small (cf. [5, §4.1]). Let $\mathfrak{c}$ be an ideal of O prime to p. Let $\mathbf{c} \in (\mathbf{A}_{\mathcal{F}}^{f(p)})^{\times}$ such that $\mathfrak{c} = \mathfrak{il}_{\mathcal{F}}(\mathbf{c})$. Here, $\mathfrak{il}_{\mathcal{F}}(\mathbf{c}) = \mathbf{c}(O \otimes \hat{\mathbf{Z}}) \cap \mathcal{F}$. We say that $\underline{\mathcal{A}} \in \mathcal{A}_K^{\Xi}(S)$ is $\mathfrak{c}$-polarised, if there exists $\lambda \in \bar{\lambda}$ (cf. $(rm2)$) such that for u as in $(c2)$, $u \in \mathbf{c}det(K)$. We can consider the subfunctors $\mathcal{E}_{\mathfrak{c},K}^{\Xi}$ and $\mathcal{C}_{\mathfrak{c},K}^{\Xi}$ of $\mathfrak{c}$-polarised quadruples. It follows that $\mathcal{E}_{\mathfrak{c},K}^{\Xi} \simeq \mathcal{C}_{\mathfrak{c},K}^{\Xi}$. This functor is represented by geometrically irreducible scheme $Sh_K^{\Xi}(\mathfrak{c})(G, X)_{/\mathbf{Z}_{(\Xi)}}$. Moreover,

$$Sh_K^{\Xi}(G, X) = \bigsqcup_{[\mathfrak{c}] \in Cl_{\mathcal{F}}^{+}(K)} Sh_K^{\Xi}(\mathfrak{c})(G, X). \tag{2.2}$$

Here, $Cl_{\mathcal{F}}^{+}(K)$ is the narrow ray class group of $\mathcal{F}$ of level $det(K)$. Following the previous notation, let $\underline{\mathcal{A}}_{\mathfrak{c},K,univ}^{\Xi}$ be the corresponding universal object.

For $g \in G(\mathbf{A}^f)$, $(A, \bar{\lambda}, \iota, \bar{\eta}) \mapsto (A, \bar{\lambda}, \iota, \overline{\eta \circ g})$ induces a right action of $G(\mathbf{A}^f)$ on $Sh(G, X)$. Let $\mathcal{G} = \mathcal{G}(G, X) = \{g \in G(\mathbf{A}) | det(g) \in \mathbf{A}^{\times}\overline{\mathcal{F}^{\times}\mathcal{F}_{\infty,+}^{\times}} / \overline{\mathcal{F}^{\times}\mathcal{F}_{\infty,+}^{\times}}\}$ and $\overline{\mathcal{E}}(G, X) = \mathcal{G}(G, X)/\overline{Z(\mathbf{Q})G(\mathbf{R})_{+}}$.

Theorem 2.3. *(Shimura) The group $\bar{\mathcal{E}}(G,X)$ is the stabiliser of $Sh(\mathfrak{c})(G,X)$ in $G(\mathbf{A})/\overline{Z(\mathbf{Q})G(\mathbf{R})_+}$ (cf. [5, Thm. 4.14]).*

When $g \in \bar{\mathcal{E}}(G,X)$ is regarded as an automorphism of $\mathcal{O}_{Sh(\mathfrak{c})(G,X)}$, we sometime write it as $\tau(g)$.

2.3. **CM points.** In this subsection, we recall the notion of a CM point on the Hilbert modular Shimura variety.

Recall, $X = (\mathbf{C} - \mathbf{R})^I = \mathcal{F} \otimes \mathbf{C}$.

Definition 2.4. A point $x = [z, g] \in Sh(G,X)(\mathbf{C})$ is said to be a CM point if $z \in X$ generates a totally imaginary quadratic extension $\mathcal{K}_x/\mathcal{F}$.

Let c_x be the complex conjugation of $\mathcal{K}_x/\mathcal{F}$. Let $\mathcal{O} = \mathcal{O}_{\mathcal{K}_x}$ be the ring of integers of $\mathcal{K}_x$. Let $T = \mathrm{Res}_{O_{(p)}/\mathbf{Z}_{(p)}} \mathbb{G}_m$, $T_x = \mathrm{Res}_{\mathcal{O}_{(p)}/\mathbf{Z}_{(p)}} \mathbb{G}_m$. The inclusion $O \hookrightarrow \mathcal{O}$ induces an inclusion $T \hookrightarrow T_x$ of $\mathbf{Z}_{(p)}$-tori. Consider the $\mathbf{Z}_{(p)}$-torus $\mathcal{T} = T/T_x$. As explained in [6, §3.2], x gives rise to the morphism

$$\widehat{\rho}_x : T_x \to G_{/\mathbf{A}^f} \tag{2.3}$$

of $\mathbf{A}^f$-group schemes. It also gives rise to a CM type Σ_x of $\mathcal{K}_x$ (cf. [6, §3.2]).

Now, suppose that (ord) is satisfied for $\mathcal{K}_x/\mathcal{F}$. Also, suppose that Σ_x is a p-adic CM type. Let $\mathbf{p} = \prod_{v \in \Sigma_{x,p}} \mathfrak{p}_v$. Consider, $\mathcal{O}_{(p)}^\times \to \mathcal{O}_{\mathbf{p}}^\times$ given by $\alpha \mapsto \alpha^{1-c_x}$. It induces an injective homomorphism

$$\mathcal{T}(\mathbf{Z}_{(p)}) \to T(\mathbf{Z}_p). \tag{2.4}$$

Let $\underline{\mathcal{A}}_x$ be the fiber of $\underline{\mathcal{A}}_{univ}$ at the geometric point x. The condition that x is a CM point translates in geometric terms as A_x is a CM abelian variety with CM type $(\mathcal{K}_x, \Sigma_x)$ (cf. [13]).

2.4. **Igusa tower.** In this subsection, we recall the notion of an Igusa tower over Hilbert modular Shimura variety. We basically add p-power level structure to the moduli problems $\mathcal{E}_K^{(p)}$ and $\mathcal{C}_K^{(p)}$.

Consider the fibered category $\mathcal{A}_{K,n}^{(p)}/SCH_{/\mathbf{Z}_{(p)}}$ defined as follows. Let $S/\mathbf{Z}_{(p)}$ be a locally Noetherian and connected scheme. The objects are the pairs $(\underline{\mathcal{A}}, j_n)_{/S}$, where $\underline{\mathcal{A}} \in \underline{\mathcal{A}}_{K^{(p)}}^{(p)}(S)$ and

$$j_n : O^* \otimes \mu_{p^n} \hookrightarrow A[p^n]$$

is a monomorphism of O-group schemes.

We define

(mor') $Mor_{\mathcal{A}_{K,n}^{(p)}}((\underline{\mathcal{A}}, j_n), (\underline{\mathcal{A}}', j_n')) := \{f \in Mor_{\mathcal{A}_{K^{(p)}}^{(p)}}(\underline{\mathcal{A}}, \underline{\mathcal{A}}') | f j_n = j_n'\}$.

Consider the functor

$$\mathcal{E}_{K,n}^{(p)} : SCH/\mathbf{Z}_{(p)} \to SETS$$
$$S \mapsto \{(\underline{\mathcal{A}}, j_n) \in \mathcal{A}_{K,n}^{(p)}(S)\}/\sim .$$

Considering $\mathcal{C}_K^{(p)}$ instead of $\mathcal{E}_K^{(p)}$ and $\simeq$ instead of $\sim$, we get a functor $\mathcal{C}_{K,n}^{(p)}$. It follows that $\mathcal{E}_{K,n}^{(p)} \simeq \mathcal{C}_{K,n}^{(p)}$.

Theorem 2.5. *The functor $\mathcal{E}_{K,n}^{(p)}$ is represented say by $I_{K,n}/\mathbf{Z}_{(p)}$ (cf. [5, §4.2.4]).*

For $n \geq m$, we have the projection morphism $\pi_{n,m} : I_{K,n} \to I_{K,m}$ induced by $O^* \otimes \mu_{p^m} \hookrightarrow O^* \otimes \mu_{p^n}$. Let $I_K = \varprojlim I_{K,n}$. Note that

$$\exists f : O^* \otimes \mu_{p^\infty} \hookrightarrow A[p^\infty] \iff \exists \widehat{f} : O^* \otimes \widehat{\mathbb{G}}_m \simeq \widehat{A}, \tag{2.5}$$

where $\widehat{A}$ is the formal completion of A along the identity section.

(act) Thus, $\mathrm{Aut}(O^* \otimes \widehat{\mathbb{G}}_m)$ acts on I_K.

Let A be a p-adic algebra and $Ig_{K/A} = \varinjlim_m \varprojlim_n I_{K,n/A/p^m A}$. In other words, Ig_K is the formal completion of I_K along the mod p fiber. As before,

(act') $\mathrm{Aut}(O^* \otimes \widehat{\mathbb{G}}_m)$ acts on Ig_K.

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$ and $W(\mathbb{F})$ be the corresponding Witt ring. Clearly,

$$Ig_{K/W(\mathbb{F})} \otimes \mathbb{F} = I_{K/\mathbb{F}}. \tag{2.6}$$

We consider a similar subfunctor $\mathcal{E}^{(p)}_{K,n}(\mathfrak{c})$ as in (2.2). It is represented by geometrically irreducible scheme $I_{K,n}(\mathfrak{c})$. We have a similar decomposition as in (2.2). We can put $I_{K,n}(\mathfrak{c})$ in the previous discussion of this subsection. Thus, we get a geometrically connected formal scheme $Ig_K(\mathfrak{c})$.

Note that $(\underline{\mathcal{A}}, j_n) \mapsto \underline{\mathcal{A}}$ induces an *étale* morphism $\pi_{K,n}(\mathfrak{c}) : I_{K,n}(\mathfrak{c}) \to Sh^{(p)}_K(\mathfrak{c})(G,X)$. Thus, we get an *étale* morphism $\pi_K(\mathfrak{c}) : I_K(\mathfrak{c}) \to Sh^{(p)}_K(G,X)$. In particular, for $x \in I_K(\mathfrak{c})$

$$\widehat{\mathcal{O}}_{I_K(\mathfrak{c}),x} \simeq \widehat{\mathcal{O}}_{Sh^{(p)}_K(G,X),\pi(x)}. \tag{2.7}$$

2.5. **Tate objects.** In this subsection, we recall some notation regarding Tate objects on the Hilbert modular Shimura variety. Basic references for this subsection are [10, §1.1] and [5, §4.1.5].

Let $\mathfrak{L}$ be a set of d linearly independent elements $l \in \mathrm{Hom}(\mathcal{F}, \mathbf{Q})$ such that $l(\mathcal{F}_+) > 0$. Let L be a lattice in $\mathcal{F}$ and n be a positive integer, we define $L_{\mathfrak{L},n} = \left\{x \in L | l(x) > -n, \forall l \in \mathfrak{L}\right\}$ and $A((L,\mathfrak{L})) = \varinjlim A[\![L_{\mathfrak{L},n}]\!]$. Pick two fractional ideals $\mathfrak{a}, \mathfrak{b}$ of O prime-to-p. To this pair, Mumford associated a certain abelian variety with real multiplication $Tate_{\mathfrak{a},\mathfrak{b}}(q)_{/\mathbf{Z}((\mathfrak{ab},\mathfrak{L}))}$ endowed with a canonical O-action ι_{can}. Formally, $Tate_{\mathfrak{a},\mathfrak{b}}(q) = \mathfrak{a}^* \otimes_{\mathbf{Z}} \mathbb{G}_m / q^{\mathfrak{b}}$. It is also endowed with a canonical polarisation λ, p^∞ level structure j_{can} and a generator ω_{can} of $\Omega_{Tate_{\mathfrak{a},\mathfrak{b}}(q)}$.

Let $\underline{\mathcal{A}}_{Tate_{\mathfrak{a},\mathfrak{b}}(q)}$ denote $(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, \iota_{can}, j_{can})$.

2.6. **Deformation theory of an ordinary abelian variety.** In this subsection, we briefly recall Serre-Tate deformation theory of an ordinary abelian variety.

Recall, $W = W(\mathbb{F})$. Let CL_W be the category of complete local W-algebras with residue field $\mathbb{F}$. Let $x = (\underline{\mathcal{A}}, j_\infty) \in I_K(\mathfrak{c})(\mathbb{F}) = Ig_K(\mathfrak{c})(\mathbb{F})$. Consider the deformation functor

$$\widehat{\mathcal{P}}_x : CL_W \to SETS$$
$$S \mapsto \{y \in I_K(\mathfrak{c})(S) | y \otimes \mathbb{F} \simeq x\}/\simeq.$$

Let T_1 be the torus $\mathfrak{d}^{-1} \otimes_{\mathbf{Z}} \mu_{p^\infty}$. The corresponding formal torus $\widehat{T}_1$ is $\mathfrak{d}^{-1} \otimes_{\mathbf{Z}} \widehat{\mathbb{G}}_m$. Note that

$$\mathcal{O}_{\widehat{T}_1} \simeq W[\![t^{\xi_1} - 1, ..., t^{\xi_d} - 1]\!], \tag{2.8}$$

for a basis $\{\xi_1, ..., \xi_d\}$ of $O/\mathbf{Z}$ and the co-ordinate t of $\mathfrak{d}^{-1} \otimes_{\mathbf{Z}} \widehat{\mathbb{G}}_m$.

Theorem 2.6. *(Serre-Tate) The deformation functor $\widehat{\mathcal{P}}_x$ is represented by the formal scheme $\widehat{S}_x = \mathrm{Spf}(\widehat{\mathcal{O}}_{I_K(\mathfrak{c}),x}) = \mathrm{Spf}(\hat{\mathcal{O}}_{Sh_K^{(p)}(\mathfrak{c}),\pi_K(x)})$. Moreover, the level p^∞-structure j_∞ induces a canonical isomorphism $\widehat{S}_x \simeq \widehat{T}_{1/W}$ (cf. [6, §2.4]).*

Let $x_{ST} = (\underline{\mathcal{A}}_{x,ST}, j_{x,ST})$ be the corresponding universal object.

2.7. Geometric Hilbert modular forms.

In this subsection, we recall the geometric definitions of classical, p-adic and mod p Hilbert modular forms.

2.7.1. *Classical Hilbert modular forms.*

In this part of subsection, we recall the geoemetric definition of classical Hilbert modular forms.

Let R be a $\mathbf{Z}_{(p)}$-algebra. A classical Hilbert modular form of polarisation ideal $\mathfrak{c}$, level K over R is a function f of isomorphism classes of $x = (\underline{\mathcal{A}}, \omega)$ where $\underline{\mathcal{A}} \in Sh_K^{(p)}(\mathfrak{c})(S)$ and ω is a differential form generating $H^0(A, \Omega_{A/S})$ over $O \otimes_{\mathbf{Z}} S$ for an R-algebra S such that the following conditions are satisfied.

$(Gc1)$ If $x \simeq x'$, then $f(x) = f(x') \in S$.
$(Gc2)$ $f(x \otimes S') = \rho(f(x))$ for any R-algebra homomorphism $\rho : S \to S'$.
$(Gc3)$ $f(\underline{\mathcal{A}}_{Tate_{\mathfrak{a},\mathfrak{b}}(q)}, \bar{\eta}^{(p)}, \omega_{can}) \in R[\![\mathfrak{ab}_{\geq 0}]\!]$ for any level $K^{(p)}$-structure $\eta^{(p)}$ of $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ defined over R, where $\mathfrak{ab}_{\geq 0} = (\mathfrak{ab} \cap \mathcal{F}_+) \cup \{0\}$.

Let $M(\mathfrak{c}, K, R)$ be the space of f's satisfying the above conditions (Gc1-3). Let $\mathbb{T} = Res_{O/\mathbf{Z}}\mathbb{G}_m$. We can define the notion of a weight $\kappa \in Mor(T, \mathbb{G}_m)$ of $f \in M(\mathfrak{c}, K, R)$ (cf. [7, §4.1]). Here Mor denotes the group scheme homomorphisms. Let $M(\kappa, \mathfrak{c}, K, R)$ denote weight κ elements in $M(\mathfrak{c}, K, R)$. It turns out that

$$M(\mathfrak{c}, K, R) = \bigoplus_{\kappa} M(\kappa, \mathfrak{c}, K, R). \tag{2.9}$$

For $f \in M(\mathfrak{c}, K, R)$, we have the following fundamental q-expansion principle (cf. [5, Thm. 4.21]).

(q-exp) For any level $K^{(p)}$-strucutre $\eta^{(p)}$ of $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ defined over R, $f \mapsto f(\underline{\mathcal{A}}_{Tate_{\mathfrak{a},\mathfrak{b}}(q)}, \bar{\eta}^{(p)}, \omega_{can}) \in R[\![\mathfrak{ab}_{\geq 0}]\!]$ determines f uniquely.

2.7.2. *p-adic Hilbert modular forms.*

In this part of subsection, we recall the geoemetric definition of p-adic Hilbert modular forms.

Let R be a p-adic algebra. A p-adic Hilbert modular form of polarisation ideal $\mathfrak{c}$, level K over R is a function f of isomorphism classes of $x = (\underline{\mathcal{A}}, j_\infty) \in I_K(\mathfrak{c})(Spec(S)) = Ig_K(\mathfrak{c})(Spf(S))$ defined over any p-adic R-algebra S such that the following conditions are satisfied.

$(Gp1)$ If $x \simeq x'$, then $f(x) = f(x') \in S$.
$(Gp2)$ $f(x \otimes S') = \rho(f(x))$ for any p-adic R-algebra homomorphism $\rho : S \to S'$.
$(Gp3)$ $f(\underline{\mathcal{A}}_{Tate_{\mathfrak{a},\mathfrak{b}}(q)}, \bar{\eta}^{(p)}) \in R[\![\mathfrak{ab}_{\geq 0}]\!]$ for any level $K^{(p)}$-strucutre $\eta^{(p)}$ of $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ defined over R.

In other words, p-adic Hilbert modular forms are the formal functions on $Ig_K(\mathfrak{c})/R$. Let $V(\mathfrak{c}, K, R)$ be the space of f's satisfying the above conditions (Gp1-3). Note that

$$V(\mathfrak{c}, K, R) = H^0(Ig_K(\mathfrak{c})_{/R}, \mathcal{O}_{Ig_K(\mathfrak{c})_{/R}}). \tag{2.10}$$

We can embed $M(\mathfrak{c}, K, R)$ in $V(\mathfrak{c}, K, R)$ as follows. Take $f \in M(\mathfrak{c}, K, R)$ and $(\underline{\mathcal{A}}, j_\infty) \in Ig_K(\mathfrak{c})(S)$. Note, j_∞ induces as isomorphism $j_\infty : O^* \otimes_{\mathbf{Z}_p} Lie(\widehat{\mathbb{G}}_m) \to Lie(A)$. Thus, $j_{\infty *}(\frac{dt}{t})$ generates $H^0(A, \Omega_A)$ as $O \otimes_{\mathbf{Z}} R$-module. Now, $f \mapsto \hat{f}(\underline{\mathcal{A}}, j_\infty) := f(\underline{\mathcal{A}}, j_{\infty *}(\frac{dt}{t}))$ gives the desired embedding. In fact, $M(\mathfrak{c}, K, R)$ is dense in $V(\mathfrak{c}, K, R)$ (cf. [5, Cor 8.4]).

In what follows, let $f \in V(\mathfrak{c}, K, R)$ and $\kappa \in Aut(O^* \otimes \widehat{\mathbb{G}}_m)$.

(wt) We say that f has weight κ if $f(\underline{\mathcal{A}}, aj_\infty) = \kappa(a)^{-1} f(\underline{\mathcal{A}}, j_\infty)$ for all $a \in O^* \otimes \widehat{\mathbb{G}}_m(S)$. Here, a acts on j_∞ in view of (2.5).

Let $V(\kappa, \mathfrak{c}, K, R)$ be the elements in $V(\mathfrak{c}, K, R)$ of weight κ.

For $f \in V(\mathfrak{c}, K, R)$, we have the following fundamental q-expansion principle (cf. [5, Thm. 4.21])

(q-exp)' For any level $K^{(p)}$-strucutre $\eta^{(p)}$ of $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ defined over R, $f \mapsto f(\underline{\mathcal{A}}_{Tate_{\mathfrak{a},\mathfrak{b}}(q)}, \bar{\eta}^{(p)}) \in R[\![\mathfrak{ab}_{\geq 0}]\!]$ determines f uniquely.

Now, let x be as in §2.6 and suppose $R \in CL_W$. We also have the following fundamental t-expansion principle (cf. [5, §8.4])

(t-exp) $f \mapsto f(x_{ST}) \in R[\![t^{\xi_1} - 1, ..., t^{\xi_d} - 1]\!]$ characterises f uniquely.

We call $f(x_{ST})$ as the t-expansion of f around x.

2.7.3. *Mod p Hilbert modular forms.* In this part of subsection, we recall geoemetric definition of mod p Hilbert modular forms. For the basic theory of mod p Hilbert modular forms, we refer the reader to [3].

We define mod p Hilbert modular forms in the same way as p-adic Hilbert modular forms, just by replacing p-adic algebras in §2.7.2 by characteristic p-algebras. Let $V(\mathfrak{c}, K, \mathbb{F})$ be the space of mod p Hilbert modular forms over $\mathbb{F}$. Note that

$$V(\mathfrak{c}, K, \mathbb{F}) = H^0(I_{K_{/\mathbb{F}}}, \mathcal{O}_{I_{K_{/\mathbb{F}}}}) \tag{2.11}$$

This follows from (2.10) and (2.6). We say that a mod p Hilbert modular form is classical if it is a reduction of a classical Hilbert modular form mod p. Let 1 denote the constant mod p Hilbert modular form 1.

Proceeding as in §2.7.2, we can define the notion of a weight for $f \in V(\mathfrak{c}, K, \mathbb{F})$. In this case, a weight κ turns out to be an element in $Mor(\mathbb{T}(\mathbb{F}_p), \mathbb{F}^\times)$. Here, $\mathbb{F}_p$ is a finite field with p elements and Mor just denotes the group homomorphisms.

In this case, we also have an appropriate analogue of (q-exp).

In view of (q-exp), $M(\mathfrak{c}, K, W)$ canonically embeds in $W[\![\mathfrak{ab}_{\geq 0}]\!]$. It turns out that the natural map from $(M(\mathfrak{c}, K, W) \otimes_W \mathbb{F}) \cap \mathbb{F}[\![\mathfrak{ab}_{\geq 0}]\!]$ to $V(\mathfrak{c}, K, \mathbb{F})$ is surjective (cf. [3]).

2.8. **Linear independence.** In this subsection, we recall a result on the linear independence of mod p Hilbert modular form and its image under certain transcendental automorphisms of the deformation space $\widehat{S}_x$ due to Hida.

Let x be as in §2.6. It comes from a CM point with CM type $(\mathcal{K}_x, \Sigma_x)$ (cf. [6, §3.2]). We write ρ for $\widehat{\rho}_x$ as in (2.3). We denote $I_{K/\mathbb{F}}$ by $Ig_{K/\mathbb{F}}$. In view of (2.6), this is notationally consistent.

As p in unramified in $\mathcal{F}/\mathbf{Q}$, $\mathfrak{d}_p^{-1} = O_p$. Thus,

$$\widehat{S}_x = \mathfrak{d}^{-1} \otimes_{\mathbf{Z}} \widehat{\mathbb{G}}_m = \mathfrak{d}_p^{-1} \otimes_{\mathbf{Z}_p} \widehat{\mathbb{G}}_m = O_p \otimes_{\mathbf{Z}_p} \widehat{\mathbb{G}}_m. \tag{2.12}$$

(aut) So, $\mathrm{Aut}(\widehat{S}_x) = O_p^\times$.

For each open compact subgroup $K \subset G(\mathbf{A}^f)$ is maximal at p, let $V_{K_{/\mathbb{F}}}$ be the geometrically irreducible component containing $\pi_K(\mathfrak{c})(x)$ in $Sh^{(p)}_{K/\mathbb{F}}$. Consider $V_{/\mathbb{F}} = \varprojlim V_{K/\mathbb{F}}$. Let $Ig(\mathfrak{c})_{/\mathbb{F}}$ be the Igusa tower over $V_{/\mathbb{F}}$. From (2.7), $\widehat{\mathcal{O}}_{Ig(\mathfrak{c}),x} \simeq \widehat{\mathcal{O}}_{V,\pi(x)}$.

(inc) Note that $\mathcal{O}_{Ig(\mathfrak{c}),x/\mathbb{F}} = \varinjlim \mathcal{O}_{Ig(\mathfrak{c}),K,x/\mathbb{F}}$ and $\mathcal{O}_{Ig(\mathfrak{c}),x}$ is dense in $\mathcal{O}_{\widehat{S}_x}$.

As shown in [6, Lem. 3.3], $\rho(\mathcal{T}(\mathbf{Z}_{(p)})) \simeq \mathcal{O}_{(p)}^\times$ in $\overline{\mathcal{E}}(G,X)$ fixes x. Thus, $\mathcal{O}_{(p)}^\times$ acts on $\mathcal{O}_{Ig(\mathfrak{c}),x}$ and $\mathcal{O}_{V,\pi(x)}$.

Recall, $\mathcal{T}(\mathbf{Z}_{(p)}) \hookrightarrow T(\mathbf{Z}_p) = O_p^\times$ (cf. (2.4)). The above action of $\mathcal{O}_{(p)}^\times$ extends to its p-adic completion $O_p^\times = Aut(\widehat{S}_x)$ (cf. (aut)). Note that $a \in O_p^\times$ acts on $\widehat{S}_x$ by $t \mapsto t^a$ where t is the canonical Serre-Tate co-ordinate.

Theorem 2.7. *(Hida) For $1 \leq i \leq n$, let $a_i \in \mathcal{T}(\mathbf{Z}_p)$ such that $a_i a_j^{-1} \notin \mathcal{T}(\mathbf{Z}_{(p)})$ for all $i \neq j$. Then, $a_i(\mathcal{O}_{Ig(\mathfrak{c}),K,x/\mathbb{F}})$ are linearly disjoint over $\mathbb{F}$ in $\widehat{\mathcal{O}}_{Ig(\mathfrak{c}),x/\mathbb{F}}$.*

Proof. For $K = 1$, this is [6, Thm. 3.19]. Thus, for K maximal at p, we are done from (inc) and the fact that the projection $\pi_K : Ig(\mathfrak{c}) \to Ig_K(\mathfrak{c})$ is *étale*. $\square$

Let f be a mod p Hilbert modular form coming from the structure sheaf of $Ig_K(\mathfrak{c})_{/\mathbb{F}}$ (cf. (2.12)). For $a \in \mathcal{T}(\mathbf{Z}_p)$, $a(f) \in \mathcal{O}_{\widehat{S}_x}$ (cf. (inc)).

Corollary 2.8. *For $1 \leq i \leq n$, let $a_i \in T_x(\mathbf{Z}_p)$ such that $a_i a_j^{-1} \notin T_x(\mathbf{Q})$, for $i \neq j$. Let J be a subset of these indices. If $\{1, f_j \in V(\mathfrak{c}, K, \mathbb{F})\}_j$ are linearly independent over $\mathbb{F}$, then $\{a_i(f_j)\}_{i,j}$ are linearly independent over $\mathbb{F}$.*

Proof. In view of (2.11), we are done by Theorem 2.7. $\square$

3. Cyclotomic derivative

In this section, we obtain an expression for the cyclotomic derivative of Katz p-adic L-function in terms of the t-expansion of certain p-adic Hilbert modular forms around a well chosen CM point x (cf. (1.8)).

Let χ be a Hecke character of infinity type $k\Sigma + \kappa(1-c)$, where $k \geq 1$ and $\kappa = \sum \kappa_\sigma \sigma \in \mathbf{Z}[\Sigma]$ with $\kappa_\sigma \geq 0$. In §3.1-3.4, we recall Hsieh's construction of a special Eisenstein series $\mathbb{E}_\chi^h$ used to compute $L_{\Sigma,\chi}^-$ and the formula of its q-expansion without proofs. In §3.5, we express $L'_{\Sigma,\lambda}$ in terms of the t-expansion of certain p-adic Hilbert modular forms around x upto an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$ (cf. (1.8)).

3.1. **Eisenstein series on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$.** In this subsection, we briefly recall the construction of an Eisenstein series on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$ in terms of a section.

Let χ be a Hecke character of infinity type $k\Sigma$, where $k \geq 1$ and $\kappa = \sum \kappa_\sigma \sigma \in \mathbf{Z}[\Sigma]$ with $\kappa_\sigma \geq 0$.

We will identify the CM-type $\Sigma \subset \mathrm{Hom}(\mathcal{K}, \mathbf{C})$ with the set $\mathrm{Hom}(\mathcal{F}, \mathbf{R})$ of archimedean places of $\mathcal{F}$ by the restriction map. Let $K_\infty^0 := \prod_{\sigma \in \Sigma} \mathrm{SO}(2, \mathbf{R})$ be a maximal compact subgroup of $\mathrm{GL}_2(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R})$. We put

$$\chi^* = \chi|\cdot|_{\mathbf{A}_{\mathcal{K}}}^{-\frac{1}{2}} \text{ and } \chi_+ = \chi|_{\mathbf{A}_{\mathcal{F}}^\times}.$$

For $s \in \mathbf{C}$, we let $I(s, \chi_+)$ denote the space consisting of smooth and K_∞^0-finite functions $\phi : \mathrm{GL}_2(\mathbf{A}_{\mathcal{F}}) \to \mathbf{C}$ such that

$$\phi(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} g) = \chi_+^{-1}(d) \left|\frac{a}{d}\right|_{\mathbf{A}_{\mathcal{F}}}^s \phi(g).$$

Conventionally, the functions in $I(s, \chi_+)$ are called *sections*. Let B be the upper triangular subgroup of GL_2. The adelic Eisenstein series associated to a section $\phi \in I(s, \chi_+)$ is defined by

$$E_{\mathbf{A}}(g, \phi) = \sum_{\gamma \in B(\mathcal{F}) \backslash \mathrm{GL}_2(\mathcal{F})} \phi(\gamma g).$$

It is known that the series $E_{\mathbf{A}}(g, \phi)$ is absolutely convergent for $\Re s \gg 0$.

3.2. **Fourier coefficients of Eisenstein series.** In this subsection, we recall the formula for the Fourier coefficients of the Eisenstein series on $\mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$.

Put $\mathbf{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let v be a place of $\mathcal{F}$ and let $I_v(s, \chi_+)$ be the local constitute of $I(s, \chi_+)$ at v. For $\phi_v \in I_v(s, \chi_+)$ and $\beta \in \mathcal{F}_v$, we recall that the β-th local Whittaker integral $W_\beta(\phi_v, g_v)$ is defined by

$$W_\beta(\phi_v, g_v) = \int_{\mathcal{F}_v} \phi_v(\mathbf{w} \begin{bmatrix} 1 & x_v \\ 0 & 1 \end{bmatrix} g_v) \psi(-\beta x_v) dx_v,$$

and the intertwining operator $M_{\mathbf{w}}$ is defined by

$$M_{\mathbf{w}} \phi_v(g_v) = \int_{\mathcal{F}_v} \phi_v(\mathbf{w} \begin{bmatrix} 1 & x_v \\ 0 & 1 \end{bmatrix} g_v) dx_v.$$

Here, dx_v is Lebesgue measure if $\mathcal{F}_v = \mathbf{R}$ and is the Haar measure on $\mathcal{F}_v$ normalized so that $\mathrm{vol}(\mathcal{O}_{\mathcal{F}_v}, dx_v) = 1$ if $\mathcal{F}_v$ is non-archimedean. By definition, $M_{\mathbf{w}} \phi_v(g_v)$ is the 0-th local Whittaker integral. It is well known that local Whittaker integrals converge absolutely for $\Re s \gg 0$, and have meromorphic continuation to all $s \in \mathbf{C}$.

If $\phi = \otimes_v \phi_v$ is a decomposable section, then it is well known that $E_{\mathbf{A}}(g, \phi)$ has the following Fourier expansion:

$$\begin{aligned} E_{\mathbf{A}}(g, \phi) &= \phi(g) + M_{\mathbf{w}} \phi(g) + \sum_{\beta \in \mathcal{F}} W_\beta(E_{\mathbf{A}}, g), \text{ where} \\ M_{\mathbf{w}} \phi(g) &= \frac{1}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}}} \cdot \prod_v M_{\mathbf{w}} \phi_v(g_v) \, ; \, W_\beta(E_{\mathbf{A}}, g) = \frac{1}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}}} \cdot \prod_v W_\beta(\phi_v, g_v). \end{aligned} \tag{3.1}$$

3.3. **Choice of the local sections.** In this subsection, we recall the choice of local sections in [8, §4.3] which gives rise to the Hilbert modular Eisenstein series $\mathbb{E}^h_\chi$ used to compute $L^-_{\Sigma,\chi}$.

We begin with some notation. Let v be a place of $\mathcal{F}$. Let $F = \mathcal{F}_v$ (resp. $E = \mathcal{K} \otimes_{\mathcal{F}} \mathcal{F}_v$). Denote by $z \mapsto \bar{z}$ the complex conjugation. Let $|\cdot|$ be the standard absolute values on F and let $|\cdot|_E$ be the absolute value on E given by $|z|_E := |z\bar{z}|$. Let $d_F = d_{\mathcal{F}_v}$ be a fixed generator of the different $\mathfrak{d}_{\mathcal{F}}$ of $\mathcal{F}/\mathbf{Q}$. Write χ (resp. χ_+) for χ_v (resp. $\chi_{+,v}$). If $v \in \mathbf{h}$, denote by ϖ_v a uniformizer of $\mathcal{F}_v$. For a set Y, denote by $\mathbb{I}_Y$ the characteristic function of Y.

Suppose that $\mathfrak{C}$ is the prime-to-p conductor of χ. We write $\mathfrak{C} = \mathfrak{C}^+\mathfrak{C}^-$ such that $\mathfrak{C}^+$ (resp. $\mathfrak{C}^-$) is a product of prime factors split (resp. non-split) over $\mathcal{F}$. We further decompose $\mathfrak{C}^+ = \mathfrak{F}\mathfrak{F}_c$ such that $(\mathfrak{F}, \mathfrak{F}_c) = 1$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$. Let $D_{\mathcal{K}/\mathcal{F}}$ be the discriminant of $\mathcal{K}/\mathcal{F}$ and let

$$\mathfrak{D} = p\mathfrak{C}\mathfrak{C}^c D_{\mathcal{K}/\mathcal{F}}.$$

Case I: $v \nmid \mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}$. We first suppose that $v = \sigma \in \Sigma$ is archimedean and $F = \mathbf{R}$. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{R})$, we put $J(g,i) := ci + d$. Define the sections $\phi^h_{k,s,\sigma}$ of weight k in $I_v(s, \chi_+)$ by

$$\phi^h_{k,s,\sigma}(g) = J(g,i)^{-k} \left|\det(g)\right|^s \cdot \left|J(g,i)\overline{J(g,i)}\right|^{-s}.$$

Suppose that v is non-archimedean. Denote by $\mathcal{S}(F)$ and (resp. $\mathcal{S}(F \oplus F)$) the space of Bruhat-Schwartz functions on F (resp. $F \oplus F$). Recall that the Fourier transform $\widehat{\varphi}$ for $\varphi \in \mathcal{S}(F)$ is defined by

$$\widehat{\varphi}(y) = \int_F \varphi(x)\psi(yx)dx.$$

For a character $\mu : F^\times \to \mathbf{C}^\times$, we define a function $\varphi_\mu \in \mathcal{S}(F)$ by

$$\varphi_\mu(x) = \mathbb{I}_{O_v^\times}(x)\mu(x).$$

If $v | p\mathfrak{F}\mathfrak{F}^c$ is split in $\mathcal{K}$, write $v = w\bar{w}$ with $w | \mathfrak{F}\Sigma_p$, and set

$$\varphi_w = \varphi_{\chi_w} \text{ and } \varphi_{\bar{w}} = \varphi_{\chi_{\bar{w}}^{-1}}.$$

To a Bruhat-Schwartz function $\Phi \in \mathcal{S}(F \oplus F)$, we can associate a Godement section $f_{\Phi,s} \in I_v(s, \chi_+)$ defined by

$$f_{\Phi,s}(g) := \left|\det g\right|^s \int_{F^\times} \Phi((0,x)g)\chi_+(x) \left|x\right|^{2s} d^\times x, \tag{3.2}$$

where $d^\times x$ is the Haar measure on $F^\times$ such that $\mathrm{vol}(\mathcal{O}_F^\times, d^\times x) = 1$. Define Godement sections by

$$\phi_{\chi,s,v} = f_{\Phi^0_v,s}, \text{ where } \Phi^0_v(x,y) = \begin{cases} \mathbb{I}_{O_v}(x)\mathbb{I}_{d_F^{-1}O_v}(y) & \cdots v \nmid \mathfrak{D}, \\ \varphi_{\bar{w}}(x)\widehat{\varphi}_w(y) & \cdots v | \mathfrak{F}\mathfrak{F}^c. \end{cases} \tag{3.3}$$

Let $u \in \mathcal{O}_F^\times$. Let $\varphi^1_{\bar{w}}$ and $\varphi^{[u]}_w \in \mathcal{S}(F)$ be the Bruhat-Schwartz functions defined by

$$\varphi^1_{\bar{w}}(x) = \mathbb{I}_{1+\varpi_v O_v}(x)\chi_{\bar{w}}^{-1}(x) \text{ and } \varphi^{[u]}_w(x) = \mathbb{I}_{u(1+\varpi_v O_v)}(x)\chi_w(x).$$

Define $\Phi^{[u]}_v \in \mathcal{S}(F \oplus F)$ by

$$\Phi^{[u]}_v(x,y) = \frac{1}{\mathrm{vol}(1+\varpi_v O_v, d^\times x)}\varphi^1_{\bar{w}}(x)\widehat{\varphi}^{[u]}_w(y) = (\left|\varpi_v\right|^{-1} - 1)\varphi^1_{\bar{w}}(x)\widehat{\varphi}^{[u]}_w(y). \tag{3.4}$$

Case II: $v | D_{\mathcal{K}/\mathcal{F}}\mathfrak{C}^-$. In this case, E is a field. We define an embedding $E \hookrightarrow M_2(F)$ by

$$a + b\delta \mapsto \begin{bmatrix} a & b\delta^2 \\ b & a \end{bmatrix}.$$

Here, δ is as in [6, $(d1)$ and $(d2)$]. Then $\mathrm{GL}_2(F) = B(F)\rho(E^\times)$. We fix a O_v-basis $\{1, \boldsymbol{\theta}_v\}$ of $\mathcal{O}_E$ such that $\boldsymbol{\theta}_v$ is a uniformizer if v is ramified and $\overline{\boldsymbol{\theta}_v} = -\boldsymbol{\theta}_v$ if $v \nmid 2$. Let $t_v = \boldsymbol{\theta}_v + \overline{\boldsymbol{\theta}_v}$ and put

$$\varsigma_v = \begin{bmatrix} d_{\mathcal{F}_v} & -2^{-1}t_v \\ 0 & d_{\mathcal{F}_v}^{-1} \end{bmatrix}.$$

Let $\phi_{\chi,s,v}$ be the smooth section in $I_v(s, \chi_+)$ defined by

$$\phi_{\chi,s,v}(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \rho(z)\varsigma_v) = L(s,\chi_v) \cdot \chi_+^{-1}(d) \left|\frac{a}{d}\right|^s \cdot \chi^{-1}(z) \quad (b \in B(F),\, z \in E^\times). \tag{3.5}$$

Here, $L(s, \chi_v)$ is the local Euler factor of χ_v.

3.4. **q-expansion of normalized Eisenstein series.** In this subsection, we recall the formula for the q-expansion coefficients of the Hilbert modular Eisenstein series $\mathbb{E}_\chi^h$ (cf. §3.3) used to compute $L_{\Sigma,\chi}^-$.

Let $\mathcal{U}_p$ be the torsion subgroup of $\mathcal{O}_{\mathcal{F}_p}^\times$. For $u = (u_v)_{v|p} \in \mathcal{U}_p$, let $\Phi_p^{[u]} = \otimes_{v|p}\Phi_v^{[u_v]}$ be the Bruhat-Schwartz function defined in (3.4). Define the section $\phi_{\chi,s}^h(\Phi_p^{[u]}) \in I(s, \chi_+)$ by

$$\phi_{\chi,s}^h(\Phi_p^{[u]}) = \bigotimes_{\sigma \in \Sigma} \phi_{k,s,\sigma}^h \bigotimes_{\substack{v \in \mathbf{h}, \\ v \nmid p}} \phi_{\chi,s,v} \bigotimes_{v|p} f_{\Phi_v^{[u_v]},s}.$$

We put

$$X^+ = \left\{\tau = (\tau_\sigma)_{\sigma\in\Sigma} \in \mathbf{C}^\Sigma \mid \operatorname{Im}\tau_\sigma > 0 \text{ for all } \sigma \in \Sigma\right\}.$$

The holomorphic Eisenstein series $\mathbb{E}_{\chi,u}^h : X^+ \times \mathrm{GL}_2(\mathbf{A}_\mathcal{F}^f) \to \mathbf{C}$ is defined by

$$\begin{gathered} \mathbb{E}_{\chi,u}^h(\tau, g_f) := \frac{\Gamma_\Sigma(k\Sigma)}{\sqrt{|D_\mathcal{F}|_\mathbf{R}}(2\pi i)^{k\Sigma}} \cdot E_\mathbf{A}\left((g_\infty, g_f), \phi_{\chi,s}^h(\Phi_p^{[u]})\right)|_{s=0} \cdot \prod_{\sigma\in\Sigma} J(g_\sigma, i)^k, \\ (g_\infty = (g_\sigma)_\sigma \in \mathrm{GL}_2(\mathcal{F} \otimes_\mathbf{Q} \mathbf{R}),\, (g_\sigma i)_{\sigma\in\Sigma} = (\tau_\sigma)_{\sigma\in\Sigma}). \end{gathered} \tag{3.6}$$

Proposition 3.1. *Let $\mathbf{c} = (\mathbf{c}_v) \in (\mathbf{A}_\mathcal{F}^f)^\times$ such that $\mathbf{c}_v = 1$ at $v|\mathfrak{D}$ and let $\mathfrak{c} = \mathbf{c}(\mathcal{O}_\mathcal{F} \otimes_\mathbf{Z} \hat{\mathbf{Z}}) \cap \mathcal{F}$. The q-expansion of $\mathbb{E}_{\chi,u}^h$ at the cusp $(O, \mathfrak{c}^{-1})$ is given by*

$$\mathbb{E}_{\chi,u}^h|_{(O,\mathfrak{c}^{-1})}(q) = \sum_{\beta\in\mathcal{F}_+} \mathbf{a}_\beta(\mathbb{E}_{\chi,u}^h, \mathfrak{c}) \cdot q^\beta.$$

The β-th Fourier coefficient $\mathbf{a}_\beta(\mathbb{E}_{\chi,u}^h, \mathfrak{c})$ is given by

$$\begin{aligned} \mathbf{a}_\beta(\mathbb{E}_{\chi,u}^h, \mathfrak{c}) =& \beta^{(k-1)\Sigma} \prod_{w|\mathfrak{F}} \chi_w(\beta)\mathbb{I}_{O_v^\times}(\beta) \prod_{w\in\Sigma_p} \chi_w(\beta)\mathbb{I}_{u_v(1+\varpi_v O_v)}(\beta) \\ &\times \prod_{v\nmid\mathfrak{D}} \left(\sum_{i=0}^{v(\mathbf{c}_v\beta)} \chi^*(\varpi_v^i)\right) \cdot \prod_{v|\mathfrak{C}^- D_{\mathcal{K}/\mathcal{F}}} L(0,\chi_v)\widetilde{A}_\beta(\chi_v), \end{aligned}$$

where

$$\begin{aligned} \widetilde{A}_\beta(\chi_v) =& \int_{\mathcal{F}_v} \chi_v^{-1}|\cdot|_E^s(x_v + \boldsymbol{\theta}_v)\psi(-d_{\mathcal{F}_v}^{-1}\beta x_v)dx_v|_{s=0} \\ :=& \lim_{n\to\infty} \int_{\varpi_v^{-n}\mathcal{O}_{\mathcal{F}_v}} \chi_v^{-1}(x_v + \boldsymbol{\theta}_v)\psi(-d_{\mathcal{F}_v}^{-1}\beta x_v)dx_v. \end{aligned} \tag{3.7}$$

Proof. This follows from (3.1) and the calculations of local Whittaker integerals of special local sections in [9, §4.3] (*cf.* [8, Prop. 4.1 and Prop. 4.4]).

□

3.5. **Cyclotomic derivative.** In this subsection, we express $L'_{\Sigma,\lambda}$ in terms of the t-expansion of certain p-adic Hilbert modular forms around x upto an automorphism of $\overline{\mathbf{Z}}_p[\![\Gamma^-]\!]$ (cf. (1.8)).

Suppose that λ is a self-dual Hecke character of type $k\Sigma$ with the root number -1 and $k > 0$.

For $a \in \mathbf{A}_K^{f(p)}$, let $\mathfrak{c}(a) = \mathfrak{c}(\mathcal{O}_\mathcal{K})N_{\mathcal{K}/\mathcal{F}}(\mathfrak{a})$. Here, $\mathfrak{a} = a(\mathcal{O}_\mathcal{K} \otimes_\mathbf{Z} \hat{\mathbf{Z}}) \cap \mathcal{K}$. Let $U_\mathcal{K} = (\mathcal{O}_\mathcal{K} \otimes_\mathbf{Z} \hat{\mathbf{Z}})^\times$, $Cl_- = \mathcal{K}^\times \mathbf{A}_{\mathcal{F},f}^\times \backslash \mathbf{A}_{\mathcal{K},f}/U_\mathcal{K}$ and Cl_-^{alg} be the subgroup of Cl_- generated by the ramified primes. Let $\mathcal{U}_p$ be the torsion subgroup of $(\mathcal{O}_\mathcal{F} \otimes_\mathbf{Z} \mathbf{Z}_p)^\times$ and let $\mathcal{U}^{alg} = U_\mathcal{K} \cap (\mathcal{K}^\times)^{1-c}$. Let $\mathcal{D}_1$ (resp. $\mathcal{D}_0$) be a set of representatives of Cl_-/Cl_-^{alg} in $(\mathbf{A}_{\mathcal{K},f}^{(\mathfrak{D})})^\times$ (resp. $\mathcal{U}_p/\mathcal{U}^{alg}$ in $\mathcal{U}_p$).

The following theorem gives a formula for $\mu(L'_{\Sigma,\lambda})$ in terms of $\mu(\mathbb{E}^h_{\lambda',u})$. Here, $\mathbb{E}^h_{\lambda',u}$ denotes the derivative of $\mathbb{E}^h_{\lambda N^k,u}$ with respect to k at $k=0$.

Theorem 3.2. *Suppose that $p \nmid h_\mathcal{K}^- \cdot D_\mathcal{F}$. Then, we have*

$$\mu(L'_{\Sigma,\lambda}) = \inf_{(u,a)\in\mathcal{D}_0\times\mathcal{D}_1} v_p\Big(\frac{\mathbf{a}_\beta(\mathbb{E}^h_{\lambda',u},\mathfrak{c}(a))}{\log_p(1+p)}\Big).$$

Proof. We follow the notation of [8, §5.2]. For a Hecke character χ and $a \in \mathcal{D}_1$, we put $\mathcal{E}_{\chi,u,a} = \mathbb{E}^h_{\chi,u}|_{\mathfrak{c}(a)}$. Sometimes, we drop χ from the notation.

The global root number being -1 implies that $\mathcal{E}_{\lambda,u,a} = 0$ (cf. Lemma 4.1). As $p \nmid h_\mathcal{K}^- D_\mathcal{F}$, from [loc. cit., remark on pp.19 and proof of Thm. 5.5], it follows that

$$\mathcal{E}_{\lambda N^k}(t) = \sharp(\mathcal{U}^{alg}) \sum_{(u,a)\in\mathcal{D}_0\times\mathcal{D}_1} \lambda N^k(a)\mathcal{E}_{\lambda N^k,u,a}|[a](t^{\langle a\rangle_\Sigma u^{-1}})$$

equals $L^-_{\Sigma,\lambda N^k}$ upto an automorphism of $\overline{\mathbf{Z}}_p[\![T_1,...,T_d]\!]$. Thus,

$$\mathcal{E}_{\lambda'}(t) = \sharp(\mathcal{U}^{alg}) \sum_{(u,a)\in\mathcal{D}_0\times\mathcal{D}_1} \lambda(a)\mathcal{E}_{\lambda',u,a}|[a](t^{\langle a\rangle_\Sigma u^{-1}}).$$

It follows from (1.4) that $L'_{\Sigma,\lambda}$ equals $\frac{\mathcal{E}_{\lambda'}(t)}{\log_p(1+p)}$ upto an automorphism of $\overline{\mathbf{Z}}_p[\![T_1,...,T_d]\!]$. Being a p-adic limit of classical eigenforms, $\mathcal{E}_{\lambda',u,a}$ is a p-adic eigenform.

Note that $p \nmid \sharp(\mathcal{U}^{alg})$. In view the linear independence of mod p Hilbert modular forms (cf. Corollary 2.7), this finishes the proof. We refer to [loc. cit., proof of Thm 5.5] for the details.

$\square$

4. Proof of Theorem A

In this section, we prove Theorem A. In §4.1, we firstly give an outline of the proof. In §4.2, under the global root number being -1 hypothesis, we prove the vanishing of the corresponding Hilbert modular Eisenstein series (cf. §3.4). In §4.3 - 4.5, we prove the Theorem.

4.1. **An outline.** In this subsection, we give an outline of the proof of Theorem A. Some of the notation used here is not followed in the rest of the section.

Let λ be a Hecke character as in §3.5. To determine $\mu(L^{'}_{\Sigma,\lambda})$, we try to study $\mathbf{a}_{\beta}(\mathbb{E}^{h}_{\lambda',u}, \mathfrak{c}(a))$ (cf. Theorem 3.2). For a given $\beta \in \mathcal{F}_{+}$, as χ varies over Hecke characters as in §3.3 with conductor being divisible by a fixed finite set of primes, then all but finitely many terms $\mathbf{a}_{\beta}(\mathbb{E}^{h}_{\chi,u}, \mathfrak{c}(a))$ equal one or zero (cf. (3.7)). We say that such a term is non-trivial, if it does not equal one or zero. As k varies, the conductor of λN^{k} is divisible by a finite set of fixed primes in $\mathcal{K}$. Say an upper bound for the number of non-trivial terms in $\mathbf{a}_{\beta}(\mathbb{E}^{h}_{\lambda N^{k},u}, \mathfrak{c}(a))$ is $n = n_{\beta}$. For $1 \leq i \leq n$, let us denote the non-trivial places by $(v_i)_i$. For a place v, let $\mathbf{a}_{\beta,\chi,v}$ denote the local factor correspoding to v. For any $\beta_v \in \mathcal{F}_{v}^{\times}$, we define $\mathbf{a}_{\beta_v,\chi,v}$ by the same expression as the one for $\beta \in \mathcal{F}_{+}$. In other words, the terms appearing in the expression for $\mathbf{a}_{\beta,\chi,v}$ are also defined for any $\beta_v \in \mathcal{F}_{v}^{\times}$. In this notation,

$$\mathbf{a}_{\beta}(\mathbb{E}^{h}_{\chi,u}, \mathfrak{c}(a)) = \prod_{i=1}^{n} \mathbf{a}_{\beta,\chi,v_i}.$$

In particular, for an integer k

$$\mathbf{a}_{\beta}(\mathbb{E}^{h}_{\lambda N^{k},u}, \mathfrak{c}(a)) = \prod_{i=1}^{n} \mathbf{a}_{\beta,\lambda N^{k},v_i}.$$

Thus, by our definition (1.6) and the Leibnitz product rule

$$\mathbf{a}_{\beta}(\mathbb{E}^{h}_{\lambda',u}, \mathfrak{c}(a)) = \sum_{i=1}^{n} (\prod_{j=1, j\neq i}^{n} \mathbf{a}_{\beta,\lambda,v_j}) \mathbf{a}_{\beta,\lambda',v_i}. \tag{4.1}$$

As the root number of λ is -1, $\mathbf{a}_{\beta}(\mathbb{E}^{h}_{\lambda,u}, \mathfrak{c}(a))$ equals zero (cf. Lemma 4.1). So, there exists at least one j such that $\mathbf{a}_{\beta,\lambda,v_j}$ equals zero. Choose one such j. Now, (4.1) simplifies as

$$\mathbf{a}_{\beta}(\mathbb{E}^{h}_{\lambda',u}, \mathfrak{c}(a)) = (\prod_{j=1, j\neq i}^{n} \mathbf{a}_{\beta,\lambda,v_j}) \mathbf{a}_{\beta,\lambda',v_i}. \tag{4.2}$$

Analysing $\inf_{\beta \in \mathcal{F}_{v}^{\times}} v_p(\mathbf{a}_{\beta,\lambda,v_j})$ and $\inf_{\beta \in \mathcal{F}_{v}^{\times}} v_p(\mathbf{a}_{\beta,\lambda',v_i})$, we get the lower bound of the equality asserted in Theorem A (cf. §4.3).

The upper bound seems to be delicate. In §4.4, we construct $\beta \in \mathcal{F}_{+}$ such that

$$v_p(\frac{a_{\beta}(\mathbb{E}^{h}_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}) = \mu_{p}^{'}(\lambda). \tag{4.3}$$

For a given v dividing $\mathfrak{C}^{-}$, in §4.5 we construct $\beta^{'} \in \mathcal{F}_{+}$ such that

$$v_p(\frac{a_{\beta'}(\mathbb{E}^{h}_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}) = \mu_{p}^{'}(\lambda_v). \tag{4.4}$$

This proves the upper bound of the equality asserted in Theorem A.

To construct $\beta \in \mathcal{F}_{+}$ satisfying (4.3) turns out to be equivalent to constructing $\beta \in \mathcal{F}_{+}$ such that the following conditions are satisfied.

- There exists a non-split place $v_1 \nmid \mathfrak{D}$ such that $\mathbf{a}_{\beta,\lambda,v_1}$ equals zero and for all $v \neq v_1$, $\mathbf{a}_{\beta,\lambda,v}$ does not equal zero.
- For $v|\mathfrak{C}^{-}$, $v_p(\mathbf{a}_{\beta,\lambda,v}) = \mu_p(\lambda_v)$.
- $v_p(\mathbf{a}_{\beta,\lambda',v_1}) = v_p(\log_p(1+p))$.
- For $v \nmid \mathfrak{C}^{-}$ and $v \neq v_1$, $v_p(\mathbf{a}_{\beta,\lambda,v}) = 0$.

To directly construct β satisfying the above properties seems difficult. So instead, we firstly construct $\beta_v \in \mathcal{F}_v^\times$ such that the following conditions are satisfied.

- There exists a non-split place $v_1 \nmid \mathfrak{D}$ such that $\mathbf{a}_{\beta_{v_1},\lambda,v_1}$ equals zero and for all $v \neq v_1$, $\mathbf{a}_{\beta_v,\lambda,v}$ does not equal zero.
- For $v|\mathfrak{C}^-$, $v_p(\mathbf{a}_{\beta_v,\lambda,v}) = \mu_p(\lambda_v)$.
- $v_p(\mathbf{a}_{\beta_{v_1},\lambda',v_1}) = v_p(\log_p(1+p))$.
- For $v \nmid \mathfrak{C}^-$ and $v \neq v_1$, $v_p(\mathbf{a}_{\beta_v,\lambda,v}) = 0$.

Then, we try to find $\beta \in \mathcal{F}_+$ such that $\mathbf{a}_{\beta,\lambda,v} = \mathbf{a}_{\beta_v,\lambda,v}$. However, it turns out that patching local β_v's to get a global β is not straightforward. Perhaps, this is not surprising as λ is self-dual with the root number -1. For this kind of patching to work, Hsieh introduced a certain epsilon dichotomy condition for each place v on β_v (cf. Lemma 4.7). In his case, the root number is 1. Depending on a place, we modify his condition to our seeting (cf. Lemma 4.6).

Summarising, it suffices to find $\beta_v \in \mathcal{F}_v^\times$ satisfying the above dotted properties and the epsilon dichiotomy condition. This is a local question. For $v \neq v_1$, the construction of such β_v is due to Hsieh. To find such β_v, Hsieh uses some input from local theta correspondence (cf. [8, Lem. 6.1]). For $v = v_1$, we find such a β_v rather directly (cf. Lemma 4.6).

To construct β' satisfying (4.4), we follows the same strategy as in the above construction of β (cf. §4.4). In this case, for $v|\mathfrak{C}^-$ we need to consider $\mathbf{a}_{\beta,\lambda',v}$. Thus, the computation is a bit more involved.

4.2. **The vanishing of an Eisenstein series.** In this subsection, under the global root number being -1 hypothesis, the vanishing of the corresponding Hilbert modular Eisenstein series (cf. §3.4) is proven.

We start with the vanishing.

Lemma 4.1. *Let λ be a self-dual Hecke character with the root number -1. Let $\mathbb{E}^h_{\lambda,u}$ be the corresponding Eisenstein series (cf. §3.4). Then, $\mathbb{E}^h_{\lambda,u} = 0$. Moreover, for a given $\beta \in \mathcal{F}_+$ coprime-to-$\mathfrak{F}$ and satisfying $\beta \in u_v(1+\varpi_v O_v)$ for all $v|p$, there exists a non-split place v_1 such that $W_\beta(\phi_{\lambda,0,v_1}, \mathbf{c_{v_1}}) = 0$.*

Proof. Recall,

$$\mathcal{E}_\lambda(t) = \sharp(\mathcal{U}^{alg}) \sum_{(u,a)\in\mathcal{D}_0\times\mathcal{D}_1} \lambda(a)\mathcal{E}_{\lambda,u,a}|[a](t^{\langle a\rangle_\Sigma u^{-1}})$$

equals $L^-_{\Sigma,\lambda}$ upto an automorphism of $\overline{\mathbf{Z}}_p[\![T_1,...,T_d]\!]$.

As the root number of λ is -1, it follows that $L^-_{\Sigma,\lambda} = 0$ (cf. Introduction). By the linear independence of mod p Hilbert modular forms (cf. Corollary 2.7) it follows that $\mathbb{E}^h_{\lambda,u} = 0$. Thus, $\mathbf{a}_\beta(\mathbb{E}^h_{\lambda,u}, \mathfrak{c}(a)) = 0$. In particular, there exists a place v_1 such that $W_\beta(\phi_{\lambda,0,v_1}, \mathbf{c_{v_1}}) = 0$.

If β is as in the last part of the lemma, then v_1 has to be non-split as for a split v, we have $W_\beta(\phi_{\lambda,0,v}, \mathbf{c_v}) \neq 0$ (cf. (3.7)).

∎

Here is a useful corollary.

Corollary 4.2. *Let $\beta \in \mathcal{F}_+$. A necessary condition for non-vanishing of $a_\beta(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))$ is that exactly one of $W_\beta(\phi_{\lambda,0,v}, \mathbf{c_v})$'s vanishes for local v's. Moreover, such a v must be non-split.*

PROOF. The first part of the corollary follows by the Leibnitz rule (cf. (3.7)).

If β is as in Lemma 4.1, the second part follows immediately by the lemma. If it not of this form, then $W_\beta(\phi_{\lambda,0,v}, \mathbf{c_v}) = 0$ for some $v|p\mathfrak{F}$ (cf. (3.7)). Thus,

$$a_\beta(\mathbb{E}^h_{\lambda N^k,u}, \mathfrak{c}(a)) = 0$$

for any k (cf. (3.7)). In particular, $a_\beta(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a)) = 0$.

□

4.3. **A lower bound.** In this subsection, we prove greater the lower bound

$$\mu(L'_{\Sigma,\lambda}) \geq \min_{v|\mathfrak{C}^-}\{\mu'_p(\lambda), \mu'_p(\lambda_v)\}$$

of the equality asserted in Theorem A.

Let v be a local place of $\mathcal{F}$ and $|.|$ be the corresponding absolute value. Sometime, we denote $\mathcal{K}_v, \mathcal{F}_v$ by $\mathcal{K}, \mathcal{F}$ respectively.

Let λ be as in §4.2 and N be the norm Hecke character. From self-duality,

$$\lambda^*|_{\mathcal{F}_v^\times} = \tau_{\mathcal{K}_v/\mathcal{F}_v}. \tag{4.5}$$

Here, $\tau_{\mathcal{K}_v/\mathcal{F}_v}$ denotes the character associated to the extension $\mathcal{K}_v/\mathcal{F}_v$.

Let λ' be the derivative of λN^k with respect to k at $k = 0$.

Lemma 4.3. *i.* $\lambda'(u\varpi_v^n) = cn(\tau_{K/F}(u)(-|\varpi_v|^n))\log_p(|\varpi_v|)$.
ii. $\lambda'^*(u\varpi_v^n) = (-1)^n cn\tau_{K/F}(u)\log_p(|\varpi_v|)$.
iii. $\log_p(\varpi_v)$ *divides* $W_\beta(\phi_{\lambda',0,v}, c_v)$ *for any local place* $v \nmid p$ *and* $\beta \in \mathcal{F}_v^\times$.

PROOF. From (4.5), for $u \in \mathcal{O}_{\mathcal{F}_v}^\times$,

$$\lambda(u\varpi_v^n) = \tau_{\mathcal{K}/\mathcal{F}}(u\varpi_v^n)|\varpi_v|^n = \tau_{\mathcal{K}/\mathcal{F}}(u)(-|\varpi_v|)^n. \tag{4.6}$$

Now, $N = |.|_{\mathbf{A}_\mathcal{K}^\times}$. Thus, $N^k(u\varpi_v^n) = |\varpi_v|^{cnk}$. Here, $c = 1$ or 2 depending on v. Fixing u and n, we think $N^k(u\varpi_v^n)$ as a function of k. Note that

$$\lambda'(u\varpi_v^n) = (\tau_{\mathcal{K}/\mathcal{F}}(u)(-|\varpi_v|)^n)\log_p(|\varpi_v|^{cn}) = cn(\tau_{K/F}(u)(-|\varpi_v|^n))\log_p(|\varpi_v|). \tag{4.7}$$

In particular, $\lambda'(u) = 0$. In any case, $\log_p(|\varpi_v|)$ divides $\lambda'(u\varpi_v^n)$. Also note that

$$\lambda'^*(u\varpi_v^n) = |\varpi_v|^{-n}\lambda'(u\varpi_v^n) = (-1)^n cn\tau_{K/F}(u)\log_p(|\varpi_v|). \tag{4.8}$$

It follows from the formulas in [9, §4.3] that $\log_p(|\varpi_v|)$ divides $W_\beta(\phi_{\lambda',0,v}, c_v)$ for $v \nmid p$. Here, $W_\beta(\phi_{\lambda',0,v}, c_v)$ denotes the p-adic derivative of $W_\beta(\phi_{\lambda N^k,0,v}, \mathbf{c_v})$ with respect to k at $k = 0$.

□

Now, we are ready to prove the lower bound.

Proposition 4.4.

$$\mu(L'_{\Sigma,\lambda}) \geq \min_{v|\mathfrak{C}^-}\{\mu'_p(\lambda), \mu'_p(\lambda_v)\}.$$

Proof. From the definition of μ and Theorem 3.2, it follows that there exists $\beta_n \in \mathcal{F}_+$ and $(u,a) \in \mathcal{D}_0 \times \mathcal{D}_1$ such that

$$\lim_n v_p\Big(\frac{a_{\beta_n}(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}\Big) = \mu(L'_{\Sigma,\lambda}). \tag{4.9}$$

In view of §4.2, we can suppose that β_n is coprime-to-$\mathfrak{F}$ and $\beta_n \in u_v(1+\varpi_v O_v)$ for all $v|p$. We can also suppose that there exists exactly one non-split place v_n such that $W_\beta(\phi_{\lambda,0,v_n}, \mathbf{c_{v_n}})$ vanishes.

We have the following two cases.

Case I - $v_n|\mathfrak{C}^-$. From (3.7), it follows that

$$\frac{a_{\beta_n}(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)} = \beta_n^{(k-1)\Sigma}\frac{W_{\beta_n}(\phi_{\lambda',0,v_n}, \mathbf{c_{v_n}})}{\log_p(1+p)} \prod_{v\neq v_n, v|\mathfrak{C}^-} W_{\beta_n}(\phi_{\lambda,0,v}, \mathbf{c_v}), \tag{4.10}$$

By Lemma 4.3 iii.,

$$v_p\Big(\frac{W_{\beta_n}(\phi_{\lambda',0,v_n}, \mathbf{c_{v_n}})}{\log_p(1+p)}\Big) \geq v_p\Big(\frac{\log_p(|\varpi_{v_n}|)}{\log_p(1+p)}\Big).$$

From [9, (4.16) and (4.17)] ,

$$v_p(W_{\beta_n}(\phi_{\lambda,0,v}, \mathbf{c_v})) \geq \mu_p(\lambda_v).$$

Thus,

$$v_p\Big(\frac{a_{\beta_n}(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}\Big) \geq \mu'_p(\lambda_{v_n}). \tag{4.11}$$

Case II - $v_n \nmid \mathfrak{C}^-$. From (3.7), it follows that

$$\frac{a_{\beta_n}(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)} = \beta_n^{(k-1)\Sigma}\frac{W_{\beta_n}(\phi_{\lambda',0,v_n}, \mathbf{c_{v_n}})}{\log_p(1+p)} \prod_{v|\mathfrak{C}^-} W_{\beta_n}(\phi_{\lambda,0,v}, \mathbf{c_v}). \tag{4.12}$$

By a similar argument as in the previous case, we conclude

$$v_p\Big(\frac{a_{\beta_n}(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}\Big) \geq \mu'_p(\lambda). \tag{4.13}$$

In either case, we get

$$v_p\Big(\frac{a_{\beta_n}(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}\Big) \geq \min_{v|\mathfrak{C}^-}\{\mu'_p(\lambda), \mu'_p(\lambda_v)\}.$$

Thus,

$$\lim_n v_p\Big(\frac{a_{\beta_n}(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}\Big) \geq \min_{v|\mathfrak{C}^-}\{\mu'_p(\lambda), \mu'_p(\lambda_v)\}. \tag{4.14}$$

□

4.4. **An upper bound I.** In this subsection, we prove an upper bound

$$\mu(L'_{\Sigma,\lambda}) \leq \mu'_p(\lambda)$$

of the equality asserted in Theorem A.

Let $\xi = 2\delta$, where δ is as in [6, $(d1)$ and $(d2)$]. We recall a lemma on the local root number of a self-dual Hecke character.

Lemma 4.5. *Let χ be a self-dual Hecke character. Then,*

$$W(\chi_v^*) = \pm\chi_v^*(\xi).$$

Moreover,
1. If v is split, then $W(\chi_v^*) = \chi_v^*(\xi)$ and
2. If v is non-split, then $W(\chi_v^*) = (-1)^{a(\chi_v^*)+v(\mathfrak{c}(R))}\chi_v^*(\xi)$, where $\mathfrak{c}(R) = \mathcal{D}_{\mathcal{F}}^{-1}(\xi\mathcal{D}_{\mathcal{K}/\mathcal{F}}^{-1})$ (cf. [11, Prop. 3.7]*).*

We start with a couple of local lemmas.

Let v_1 be a non-split place which is relatively prime to $p\mathfrak{C}\mathfrak{C}^{\mathfrak{c}}D_{\mathcal{K}/\mathcal{F}}$ such that

$$v_p\left(\frac{\log_p(|\varpi_{v_1}|)}{\log_p(1+p)}\right) = 0. \tag{4.15}$$

Lemma 4.6. *There exists an $\eta_{v_1} \in \mathcal{F}_{v_1}^{\times}$ and $\mathbf{c_{v_1}}$ such that the following conditions are satisfied.*
i.

$$W(\lambda_{v_1}^*)\tau_{\mathcal{K}_{v_1}/\mathcal{F}_{v_1}}(\eta_{v_1}) = -\lambda_{v_1}^*(\xi), \tag{4.16}$$

ii.

$$v_p\left(\frac{W_{\eta_{v_1}}(\phi_{\lambda',0,v_1}, \mathbf{c_{v_1}})}{\log_p(1+p)}\right) = 0. \tag{4.17}$$

Proof. Choose an η_{v_1} satisfying (4.16) and let $\mathbf{c_{v_1}}$ be such that $v_1(\eta_{v_1}\mathbf{c_{v_1}})$ is odd. The existence of η_{v_1} follows by by Lemma 4.5.

Note that the condition (4.16) forces $W_{\eta_{v_1}}(\phi_{\lambda,0,v_1}, \mathbf{c_{v_1}})$ to vanish (cf. [9, (4.7)]).

From (3.7), it now follows that

$$W_{\eta_{v_1}}(\phi_{\lambda',0,v_1}, \mathbf{c_{v_1}}) = |\mathcal{D}_{\mathcal{F}}|^{-1}\log_p(|\varpi_{v_1}|)\sum_{i=0}^{v_1(\eta_{v_1}\mathbf{c_{v_1}})}(-1)^i i. \tag{4.18}$$

Now, $\sum_{i=0}^{i=n}(-1)^i i$ equals $\frac{n}{2}$ if n is even and $\frac{-(n+1)}{2}$ if n is odd.

In view of Lemma 4.5, the condition (4.16) basically puts a restriction on whether $v_1(\eta_{v_1})$ is even or odd. Thus, we are done from the above formula (4.18) and the choice of v_1 (cf. (4.15)).

□

For convenience, let us state [8, Prop. 6.3] as the following lemma.

Lemma 4.7. *(Hsieh) Let $v|\mathfrak{C}^-$. There exists an $\eta_v \in \mathcal{F}_v^{\times}$ and $\mathbf{c_v}$ such that the following conditions are satisfied.*
i.

$$W(\lambda_v^*)\tau_{\mathcal{K}_v/\mathcal{F}_v}(\eta_v) = \lambda_v^*(\xi). \tag{4.19}$$

ii.

$$v_p(W_{\eta_v}(\phi_{\lambda,0,v}, \mathbf{c_v})) = \mu_p(\lambda_v). \tag{4.20}$$

With enough preparations, we have the following proposition.

Proposition 4.8. *There exists $\beta \in \mathcal{F}_+$, $u \in \mathcal{D}_0$ and $\mathfrak{c}(a)$ such that*

$$v_p(\frac{a_\beta(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}) = \mu_p^{'}(\lambda).$$

In particular,

$$\mu(L_{\Sigma,\lambda}^{'}) \leq \mu_p^{'}(\lambda).$$

Proof. As explained in §4.1, we basically modify the strategy in [9, proof of Prop. 6.7] to our setting.

Let v_1 and η_v's be as in the last two lemmas. We extend $(\eta_v)_{v=v_1, v|\mathfrak{C}^-}$ to an idele $\eta = (\eta_v)$ in $\mathbf{A}_\mathcal{F}^\times$ such that

$$W(\lambda_v^*)\tau_{\mathcal{K}_v/\mathcal{F}_v}(\eta_v) = \lambda_v^*(\xi)$$

for every finite place $v \neq v_1$. From [14],

$$W(\lambda_\sigma^*) = i^{2\kappa_\sigma+1} = \lambda_\sigma^*(\xi)$$

for $\sigma \in \Sigma$. As $W(\lambda^*) = -1$, we conclude that $\tau_{\mathcal{K}/\mathcal{F}}(\eta) = 1$. In particular, η can be written as $\beta N_{\mathcal{K}/\mathcal{F}}(a)$ for some $\beta \in \mathcal{F}_+$ and $a \in \mathbf{A}_\mathcal{K}^\times$. By the approximation theorem, a can be chosen so that $a \equiv 1 \bmod p(v_1\mathfrak{C}^-)^n$ for sufficiently large n.

Summarising, for every sufficiently small ϵ, we have $\beta \in \mathcal{F}_+^\times \cap O_{(p\mathfrak{F}\mathfrak{F}^c)}$ such that

- $|\beta - \eta_v| < \epsilon$ for all $v = v_1$ and v dividing $\mathfrak{C}^-$,
- $W(\lambda_{v_1}^*)\tau_{\mathcal{K}_{v_1}/\mathcal{F}_{v_1}}(\eta_{v_1}) = -\lambda_{v_1}^*(\xi)$ and $W(\lambda_v^*)\tau_{\mathcal{K}_v/\mathcal{F}_v}(\eta_v) = \lambda_v^*(\xi)$ for every finite place $v \neq v_1$.

Now, choose ϵ small enough so that $W_\beta(\phi_{\lambda,0,v}, \mathbf{c_v}) = W_{\eta_v}(\phi_{\lambda',0,v}, \mathbf{c_v})$ for all $v|\mathfrak{C}^-$ and
$W_\beta(\phi_{\lambda',0,v_1}, \mathbf{c_{v_1}}) = W_{\eta_{v_1}}(\phi_{\lambda',0,v_1}, \mathbf{c_{v_1}})$.

Consider, $\mathfrak{J} := \prod_{\mathfrak{q}|v_1\mathfrak{C}} \mathfrak{q}^{v_\mathfrak{q}(\beta)}$. From lemma 4.5, it follows that $v(\beta) \equiv v(\mathfrak{c}(R))$ (mod 2) for every inert place $v \nmid v_1\mathfrak{C}^-$. Thus, there exists a fractional ideal $\mathfrak{a}$ of R such that

$$\mathfrak{J} = (\beta)\mathfrak{c}(R)N_{\mathcal{K}/\mathcal{F}}(\mathfrak{a})^{-1} = (\beta)\mathfrak{c}(\mathfrak{a}). \tag{4.21}$$

Define $\mathbf{c} \in (\mathbf{A}_\mathcal{F}^f)^\times$ by $\mathbf{c}_v = \beta^{-1}$ if v is prime to $pv_1\mathfrak{C}\mathfrak{C}^c$, $\mathbf{c}_{v_1}$ as in Lemma 4.6 and $\mathbf{c}_v = 1$ otherwise. Thus, $\mathfrak{il}_\mathcal{F}(\mathbf{c}) = \mathfrak{c}(\mathfrak{a})$. Let $u \in \mathcal{U}_p$ such that $u \equiv \beta$ (mod p).

By (3.10), $a_\beta(\mathbb{E}^h_{\lambda'}, \mathfrak{c})$ equals

$$\frac{1}{|D_\mathcal{F}|_\mathbf{R}} \prod_{v \in h \smallsetminus v_1} W_{\eta_v}(\phi_{\lambda,0,v}, \mathbf{c_v}) W_{\eta_{v_1}}(\phi_{\lambda',0,v_1}, \mathbf{c_{v_1}})$$

$$= \lambda_+(\mathbf{c}) \prod_{w|\mathfrak{F}} \lambda_w(\beta) \prod_{v|\mathfrak{C}^-} W_{\eta_v}(\phi_{\lambda,0,v}, \mathbf{c_v}) W_{\eta_{v_1}}(\phi_{\lambda',0,v_1}, \mathbf{c_{v_1}}). \tag{4.22}$$

We are done by (4.17) and (4.20).

$\square$

4.5. **An upper bound II.** In this subsection, we prove an upper bound

$$\mu(L'_{\Sigma,\lambda}) \leq \mu'_p(\lambda_v)$$

of the equality asserted in Theorem A. This subsection is quite similar to the previous subsection.

Let v be a place dividing $\mathfrak{C}^-$ such that $w(\mathfrak{C}^-) = 1$.

We start with a couple of local lemmas.

Lemma 4.9. *Suppose that v is ramified. There exists an $\eta_v \in \mathcal{F}_v^\times$ and $\mathbf{c_v}$ such that the following conditions are satisfied.*
i.

$$W(\lambda_v^*)\tau_{K_v/F_v}(\eta_v) = -\lambda_v^*(\xi). \tag{4.23}$$

ii.

$$v_p(\frac{W_{\eta_v}(\phi_{\lambda',0,v}, \mathbf{c_v})}{\log_p(1+p)}) = v_p(\frac{\log_p(|\varpi_v|)}{\log_p(1+p)}). \tag{4.24}$$

Proof. Recall, the condition (4.23) just puts a condition on the parity of $v(\eta_v)$ (cf. Lemma 4.5). Start with an η_v satisfying this condition along with $v(2\eta_v) \geq -1$. Then, it follows that (cf. [9, Prop. 4.4])

$$W_{\eta_v}(\phi_{\lambda N^k,0,v}, \mathbf{c_v}) = \psi^0(t_w\eta_v)|2d_F^{-1}|((\lambda N^k)^*(\theta^{-1})|\varpi_v|^{1/2} + (\lambda N^k)^*(-2\eta_v d_{\mathcal{F}}^{-1})\epsilon(1,(\lambda N^k)_+|.|^{-1},\psi)). \tag{4.25}$$

Thus,

$$W_{\eta_v}(\phi_{\lambda',0,v}, \mathbf{c_v}) = a((\lambda')^*(\theta^{-1})|\varpi_v|^{1/2} + (\lambda')^*(-2\eta_v d_{\mathcal{F}}^{-1})\epsilon(1,\lambda_+|.|^{-1},\psi) + \lambda^*(-2\eta_v d_{\mathcal{F}}^{-1})\epsilon(1,\lambda'_+|.|^{-1},\psi)), \tag{4.26}$$

where $a = \psi^0(t_w\eta_v)|2d_{\mathcal{F}}^{-1}|$.

Now, $\lambda^*((-2\eta_v d_{\mathcal{F}}^{-1})) = \tau_{\mathcal{K}_v/\mathcal{F}_v}((-2\eta_v d_{\mathcal{F}}^{-1}))$ (cf. (4.5)). This value is already by (4.23). So, the only quantity we can vary is $(\lambda')^*(-2\eta_v d_F^{-1})$.

By (4.8), it is clear that we can choose an η_v satisfying (4.24) as well. Let $\mathbf{c_v} = \mathbf{1}$.

∎

We now consider the inert case.

Lemma 4.10. *Suppose that v is inert. There exists an $\eta_v \in F_v^\times$ and $\mathbf{c_v}$ satisfying the same conditions as of the previous lemma.*

Proof. In view of (4.23) and (4.24), it follows that the only change in this and the ramified case is the formula for $W_{\eta_v}(\phi_{\lambda N^k,0,v}, \mathbf{c_v})$. Recall, $\lambda N^k|_{O_v^\times} = 1$ (cf. (4.3)).

Let $\eta_v \in O_v$. Thus, from [9, Prop. 4.5]

$$W_{\eta_v}(\phi_{\lambda N^k,0,v}, \mathbf{c_v}) = b(-|\varpi_v| + \sum_{j=0}^{v(2\beta)} (\lambda N^k)^*(\varpi_v^j)|1-\varpi_v| - (\lambda N^k)^*(\varpi_v^{v(2\beta)+1})|\varpi_v|), \tag{4.27}$$

where $b = |d_F^{-1}|L(0,\lambda N^k)$ (cf. [loc. cit., (4.16)]).

From (4.8), $\sum$ expression is quite similar to (4.18). As the argument is very similar to the proof of Lemma

4.6, we skip the details.

□

With enough preparations, we have the following proposition.

Proposition 4.11.
$$\mu(L^{'}_{\Sigma,\lambda}) \leq \min_{v|\mathfrak{C}^-}\{\mu^{'}_p(\lambda), \mu^{'}_p(\lambda_v)\}.$$

PROOF. If $\mu_p(\lambda_v) = 0$ for all $v|\mathfrak{C}^-$, then the proposition follows from Proposition 4.8. Thus, we suppose that $\mu_p(\lambda_{v_1}) \neq 0$ for $v_1|\mathfrak{C}^-$.

In this case, $w_1(\mathfrak{C}^-) = 1$ (cf. [8, proof of Prop. 6.3]). Thus, we are in the situation of the last two lemmas.

Let η_{v_1} be as in these lemmas depending on whether v_1 is ramified or inert. Let η_v for $v|\mathfrak{C}^-$ and $v \neq v_1$ be as in Lemma 4.7.

Extend $(\eta_v)_{v|\mathfrak{C}^-}$ to an idele (η_v) in $\mathbf{A}_{\mathcal{F}}^{\times}$ in the same way as in the proof of Proposition 4.8. Proceeding as in the same proof, we get $\beta \in \mathcal{F}_+$, $u \in \mathcal{D}_0$ and $\mathfrak{c}(a)$ such that
$$v_p(\frac{a_\beta(\mathbb{E}^h_{\lambda',u}, \mathfrak{c}(a))}{\log_p(1+p)}) = \mu^{'}_p(\lambda_{v_1}).$$

□

Corollary 4.12. *Theorem A holds.*

PROOF. This follows from Proposition 4.8 and Proposition 4.11.

□

5. Non-vanishing of anticyclotomic regulator

In this section, we prove the non-vanishing of the anticyclotomic regulator of a self-dual CM modular form with the global root number -1.

In this section, we suppose that $\mathcal{F} = \mathbf{Q}$. Let the notation and hypothesis be as in Theorem A. Let f_λ be the CM modular form associated to λ.

To finish the notation, let $\mathcal{R}_\lambda$ be the regulator of the $\Lambda[\![\Gamma^-]\!]$-adic height pairing associated to f_λ (cf. [2, §4.4]).

Our application is as follows.

Proposition 5.1. *Suppose that $p \nmid h_{\mathcal{K}}$. Then, the anticyclotomic regulator $\mathcal{R}_\lambda$ does not vanish.*

PROOF. We follow the notation in [loc. cit.].

Let $\mathcal{X}^*(\mathcal{K}^-_\infty)$ be the anticyclotomic dual-Selmer group associated to $f_\lambda/\mathcal{K}^-_\infty$. It is a $\Lambda[\![\Gamma^-]\!]$-module of rank one (cf. [loc. cit., Thm. 2.2]). Let $\mathcal{X} \in \Lambda[\![\Gamma^-]\!]$ be the characteristic ideal of the torsion sub-module of $\mathcal{X}^*(\mathcal{K}^-_\infty)$.

In [2, Thm. 2.2], it is proven that

$$\mathcal{X}\mathcal{R} = (L'_{\Sigma,\lambda}) \tag{5.1}$$

as ideals of $\overline{\mathbf{Z}}_p[\![\Gamma^-]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

The proposition follows by Theorem A.

□

Remark. When λ is a Grössencharacter of a CM elliptic curve, the above proposition is proven via Iwasawa theory of CM elliptic curves and a non-vanishing result of Rohrlich (cf. [1, App.]).

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